There will be no formal discussion section. Instead, I will have office hours in the classroom.
Time and Place: 8am-9.5am at 75 Evans & 9.5am-11am at 228 Dwinelle

Practice Problems for Midterm 2

1. Write "TRUE" if the statement is always true, "FALSE" if it is sometimes false. ***No** explanations are needed.*[1]

   a. Given a subspace W of V, the orthogonal projection map from V to W is a one-to-one linear transformation.
   b. The orthogonal complement of the null space of A is the same as the column space of A if A is symmetric.
   c. If the orthogonal complement of the null space of A is the same as the column space of A, then A is symmetric.
   d. A square matrix A is invertible if and only if 0 is not an eigenvalue of A.
   e. Every orthogonal set is a linearly independent set.
   f. If A^3 is diagonalizable, then A is diagonalizable as well.
   g. If A^3 is diagonalizable, then there exists diagonalizable B such that $A^3 = B^3$.
   h. Let A be an $n \times n$ matrix. If the sum of entries in a single column is zero for all columns, 0 is an eigenvalue of A.
   i. Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are vectors in $\mathbb{R}^n$. If $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is an orthonormal set, then it is a basis for $\mathbb{R}^n$.

Answer: F, T, F, T, F, F, T, T, T.

2. Let $A = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$.

   a. Is A diagonalizable? If so, find an invertible 2×2 matrix P and a diagonal matrix D satisfying

$$P^{-1}AP = D$$

   b. Compute $D^3 - 2D^2 + D$. Then, compute $A^3 - 2A^2 + A$ using the previous result.

Answer: $P = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $D^3 - 2D^2 + D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, $A^3 - 2A^2 + A = \begin{bmatrix} -4 & -4 \\ 6 & 6 \end{bmatrix}$.

3. Let A be

$$\begin{bmatrix} 3 & -4 & -4 \\ 2 & 1 & -4 \\ -2 & 0 & 5 \end{bmatrix}$$

whose characteristic polynomial $\chi_A(\lambda)$ is $-(\lambda - 1)(\lambda - 3)(\lambda - 5)$.

   a. Find three linearly independent eigenvectors and, using them, find a diagonal matrix D and an invertible matrix P such that

$$P^{-1}AP = D$$

   b. You have found only one pair of (D, P) in problem **a.** Find all possible D's. For each D, find one corresponding invertible matrix P such that $P^{-1}AP = D$.

Answer: $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ and exchange 1, 3, 5 to get all D's. $P = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ and exchange columns accordingly.

[1] You can skip f and h. If you want to skip one more, g could also be skipped.

4. Consider

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

Note that they are orthogonal to each other and let W be the span of $\{\mathbf{u}, \mathbf{v}\}$.[2]

a. Define a linear transformation T from $\mathbb{R}^4$ to $\mathbb{R}^4$ as the orthogonal projection

$$T(\mathbf{x}) = \text{proj}_W(\mathbf{x}) = \frac{\mathbf{u} \cdot \mathbf{x}}{3}\mathbf{u} + \frac{\mathbf{v} \cdot \mathbf{x}}{3}\mathbf{v}.$$

Let us denote the matrix of the linear transformation T by $[T]$. Find eigenvalues of $[T]$.

b. Is the matrix $[T]$ diagonalizable?

Answer: 0 and 1. Yes, it is diagonalizable.

5. a. In $\mathbb{R}^5$, you are given three vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \ v_3 = \begin{bmatrix} 4 \\ -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

Apply Gram-Schmidt (orthogonalization) process to find an orthogonal basis of $\text{Span}\{v_1, v_2, v_3\}$.

b. Solve the least-squares problem

$$A\mathbf{x} = \mathbf{b} \text{ where } A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 1 & -3 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

in two different ways. [Both methods will take a bit long time.]

Answer: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis and the LS solution is $\hat{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

[2] Hint. Considering the geometric picture, you can solve this problem without finding the matrix T.