There will be no formal discussion section. Instead, I will have office hours in the classroom.
Time and Place: 8am-9.5am at 75 Evans & 9.5am-11am at 228 Dwinelle

# Practice Problems for Midterm 1

1. For a real number $c$, consider the linear system

$$\begin{aligned} x_1 + x_2 + cx_3 + x_4 &= c \\ -x_2 + x_3 + 2x_4 &= 0 \\ x_1 + 2x_2 + x_3 - x_4 &= -c \end{aligned}$$

For what $c$, does the linear system have a solution?

Answer: $c \neq 2$.

2. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ sends $\begin{bmatrix}1\\2\end{bmatrix}$ to $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}3\\4\end{bmatrix}$ to $\begin{bmatrix}3\\4\\5\end{bmatrix}$.

a. Find $T\left(\begin{bmatrix}5\\6\end{bmatrix}\right)$.

b. Find a vector $\mathbf{v} \in \mathbb{R}^2$ with $T(\mathbf{v}) = \begin{bmatrix}7\\8\\7\end{bmatrix}$, or else explain why no such vector exists.

Answer: a. $\begin{bmatrix}5\\6\\7\end{bmatrix}$, b. No such $\mathbf{v}$ exists.

3. Determine the inverse of the following matrix by two different methods:

[one method is to use Cramer's Rule, but you don't necessarily need to know this for Midterm 1]

$$A = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 4\\2 & 2 & 1\end{bmatrix}$$

Answer: $\begin{bmatrix}3 & -4 & 7\\-4 & 5 & -8\\2 & -2 & 3\end{bmatrix}$.

4. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ has the standard matrix with 2 pivots and we know

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\-1\end{bmatrix} \qquad T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Which of the following are a possible standard matrix of $T$?

a) $\begin{bmatrix}-1 & 1 & 0\\-2 & 2 & 0\\1 & -1 & 0\end{bmatrix}$ b) $\begin{bmatrix}1 & 1 & -2\\2 & 2 & -4\\1 & -1 & -2\end{bmatrix}$ c) $\begin{bmatrix}1 & 1 & -2\\0 & 2 & -2\\0 & -1 & 1\end{bmatrix}$ d) $\begin{bmatrix}1 & 1 & -2\\0 & 2 & -2\\2 & -1 & -1\end{bmatrix}$

Answer: $c$ and $d$.

5. Consider the matrices

$$A = \begin{bmatrix}1 & 0 & -1\\2 & 1 & 0\\0 & 1 & 0\\-1 & 1 & 2\end{bmatrix} \qquad B = \begin{bmatrix}1 & 0 & 1 & 0\\2 & 1 & 0 & -1\\0 & -1 & 2 & -1\end{bmatrix}$$

a. Calculate the matrix $AB$.

b. Calculate $\det(AB)$. [You are welcome to skip this problem. For others, here is some clue: $\dim \mathrm{Nul} B \geq 1$.]

Answer: a. $\begin{bmatrix}1 & 1 & -1 & 1\\4 & 1 & 2 & -1\\2 & 1 & 0 & -1\\1 & -1 & 3 & -3\end{bmatrix}$, b. 0.

6. Let a map $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined as

$$T(x, y, z) = (x - y + z, x) \text{ or equivalently } T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - y + z \\ x \end{bmatrix}$$

Show that $T$ is a **onto** linear transformation. Does $T$ map $\mathbb{R}^3$ **one-to-one** $\mathbb{R}^2$?

Answer: It is not one-to-one.

7. Suppose $A = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$ is the standard matrix for a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $B = \begin{bmatrix} 6 & 7 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$ is the standard matrix for a linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^4$.

a. Check if $A$ is an invertible matrix. (If it is, find the inverse. If not, prove why it is not invertible.)
b. Find the standard matrix for $S \circ T \circ T^{-1} \circ T$.

Answer: a. It is., b. Transpose[1] of $\begin{bmatrix} -5 & 0 & -2 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$.

8. Compute the determinant of the following matrix. [You can do it!]

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 & 10 \\ 0 & 0 & 0 & -1 & 1 \\ 3 & 3 & 7 & 1 & 2 \\ 5 & -1 & 3 & 9 & 2 \end{bmatrix}$$

Answer: 404.

9. Which of the following tranformations is a linear transformation?

a. Rotation by $\pi$ about the origin followed by reflection across the line passing through $(1, 1)$ and $(1, 2)$
b. Reflection across the line passing through $(1, 1)$ and $(-1, -1)$
c. Rotation by $\pi/2$ about $(1, 1)$ followed by reflection across the line $y = x$ followed by rotation by $\pi/2$ about $(1, 1)$

Answer: b and c.

10. Mark each statement True or False. Justify your answer precisely.

a. For a matrix $A$, there exists a unique echelon form of $A$.
b. Suppose that six vectors $v_1, v_2, \cdots, v_6$ satisfy :

$$\{v_1, v_2, v_3, v_4\},\ \{v_3, v_4, v_5, v_6\}, \text{ and } \{v_5, v_6, v_1, v_2\} \text{ are linearly independent sets of vectors.}$$

Then, $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a linearly independent set.
c. If $A$ is a $2 \times 2$ matrix such that $A^2 = 0$, then $A = 0$.
d. If $A$ is a $5 \times 5$ matrix such that $\det(2A) = 2\det(A)$, then $A = 0$.
e. If one row in some echelon form of the augmented matrix of a system is $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, then the system is inconsistent.
f. The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects points about the line $y = 1$ is linear.
g. The solution set of a consistent linear system $A\mathbf{x} = \mathbf{b}$ consists of the vectors $\mathbf{p} + \mathbf{h}$, where $\mathbf{p}$ is one particular solution of the system and $\mathbf{h}$ ranges over the solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$.
h. The columns of any $4 \times 5$ matrix are linearly dependent.
i. The columns of any $4 \times 3$ matrix are linearly independent.
j. Every line in $\mathbb{R}^3$ is a linear subspace.
k. If the two $n \times n$ matrices $A$, $B$ are invertible, then so is $AB$ and $(AB)^{-1} = A^{-1}B^{-1}$.
l. If $A$ is a $4 \times 4$ matrix and the system $A\mathbf{x} = \mathbf{e}_j$ is consistent for each $j = 1, 2, 3, 4$, then $A$ is invertible.

Answer: Only g, h, and l are True.

[1] The only reason I've written the answer in this strange way is to make solutions fit into this page.