1. Which of the following conditions makes the system $A\mathbf{x} = \mathbf{b}$, with A an 4×5 matrix, have at least one solution?

   a) Always b) Never c) When A has four pivots
   d) When A has a left nullspace e) When $\mathbf{b}$ is in Nul$A$ f) When $\mathbf{b}$ is in Col$A$

2. Which of the following is the smallest possible dimension of the null space of a 8×5 matrix A?

   a) 0 b) 1 c) 2 d) 3 e) 4

3. What is the largest possible dimension of the null space of 5×8 matrix A?

   a) 9 b) 8 c) 7 d) 6 5) 4

4. Which of the following matrices are similar to
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}?$$

   a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$ e) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

5. Which of the following conditions is necessary for $A\mathbf{x} = \mathbf{x}$ (A is an 3×3 matrix) to have at least one solution?

   a) Always b) When $\det A = 0$ c) When $\det(A - I) = 0$
   d) When $\mathbf{x}$ is in Nul$A$ e) When $\mathbf{x}$ is in Col$(A - I)$ f) When $\det A$=1

6. Which of the following sets are orthogonal?

   a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ b) $\left\{ \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right\}$ c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

   d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ e) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ -11 \\ 6 \end{bmatrix} \right\}$

7. A number λ is not an **eigenvalue** of an $n \times n$ matrix A if and only if:

   a) $\text{Nul}(A - \lambda I_n) = \{\mathbf{0}\}$ b) $\det(A - \lambda I_n) = 0$ c) $A\mathbf{x} = \lambda\mathbf{x}$ for more than two $\mathbf{x}$'s$\in \mathbb{R}^n$
   d) $A - \lambda I_n$ is not invertible e) $(A - \lambda I_n)\mathbf{x} = \mathbf{b}$ is not consistent for some $\mathbf{b} \in \mathbb{R}^n$.

8. The following is an eigenvalue of $A = \begin{bmatrix} 3 & 3 \\ 4 & 7 \end{bmatrix}$:

   a) 3 b) 5 c) 7 d) 9 e) 2

9. Which of the following matrices is similar to $\begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$?

   a) $\begin{bmatrix} -4 & 1 \\ 0 & 5 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$ c) $\begin{bmatrix} 5 & 1 \\ 0 & -4 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 6 \\ 0 & -2 \end{bmatrix}$ e) None of them

10. For some basis of $\mathbb{R}^2$, the coordinate vector of the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and that of the vector $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ is $\begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then, the coordinate vector of $\begin{bmatrix} 6 \\ 7 \end{bmatrix}$ is

    a) $\begin{bmatrix} 8 \\ 7 \end{bmatrix}$ b) $\begin{bmatrix} 8 \\ 9 \end{bmatrix}$ c) $\begin{bmatrix} 9 \\ 8 \end{bmatrix}$ d) $\begin{bmatrix} 9 \\ 9 \end{bmatrix}$ e) $\begin{bmatrix} 8 \\ 8 \end{bmatrix}$

11. Which of the following vectors belongs to a basis $\mathcal{B}$ of $\mathbb{R}^2$ when the $\mathcal{B}$-coordinate of $\begin{bmatrix}4\\9\end{bmatrix}$ is $\begin{bmatrix}2\\1\end{bmatrix}$ and that of $\begin{bmatrix}3\\7\end{bmatrix}$ is $\begin{bmatrix}1\\1\end{bmatrix}$?

    a) $\begin{bmatrix}1\\0\end{bmatrix}$ b) $\begin{bmatrix}1\\1\end{bmatrix}$ c) $\begin{bmatrix}1\\2\end{bmatrix}$ d) $\begin{bmatrix}1\\3\end{bmatrix}$ e) $\begin{bmatrix}1\\9\end{bmatrix}$

12. Under which circumstances is the square matrix A guaranteed to have non-zero determinant?

    a) A has positive entries everywhere b) A has orthonormal columns c) A has eigenvalue 1
    d) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for a $\mathbf{b} \in \mathbb{R}^n$ e) None of them

13. Which of the following upper-triangular matrices with only one eigenvalue has an one-dimensional eigenspace?

    a) $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ b) $\begin{bmatrix}2&0&0\\0&2&7\\0&0&2\end{bmatrix}$ c) $\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}$ d) $\begin{bmatrix}-1&1&0\\0&-1&0\\0&0&-1\end{bmatrix}$ e) $\begin{bmatrix}1&2&3\\0&1&2\\0&0&1\end{bmatrix}$

14. Which of the following matrices is not diagonalizable over $\mathbb{R}$?

    a) $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ b) $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ c) $\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ d) $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ e) $\begin{bmatrix}1&1\\1&1\end{bmatrix}$

15. Which of the following matrices is not diagonalizable over $\mathbb{C}$?

    a) $\begin{bmatrix}1&1\\0&1\end{bmatrix}$ b) $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ c) $\begin{bmatrix}2&-1\\1&3\end{bmatrix}$ d) $\begin{bmatrix}1&2\\6&-1\end{bmatrix}$ e) $\begin{bmatrix}1&2\\2&1\end{bmatrix}$

16. Which of the following matrices is diagonalizable over $\mathbb{C}$ but not over $\mathbb{R}$?

    a) $\begin{bmatrix}1&-2\\2&5\end{bmatrix}$ b) $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ c) $\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ d) $\begin{bmatrix}2&1\\0&2\end{bmatrix}$ e) $\begin{bmatrix}1&1\\1&1\end{bmatrix}$

17. Which of the following matrices is diagonalizable over $\mathbb{R}$ but not over $\mathbb{C}$?

    a) $\begin{bmatrix}1&3\\3&1\end{bmatrix}$ b) $\begin{bmatrix}2&1\\1&-1\end{bmatrix}$ c) $\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ d) $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ e) None of them

18. Which of the following linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{x} \mapsto A\mathbf{x}$ are given by an orthogonal matrix A?*

    a) Reflection across the line $x = y$
    b) Rotation by $\pi/4$ about the origin
    c) A shear transformation fixing the line $y = 0$
    d) Reflection across the line $x = y$ followed by reflection across the line $x = 0$
    e) Scaling by 2 followed by rotation by $\pi/4$ about the origin followed by scaling by $1/2$

19. Which of the following tranformations is a linear transformation?*

    a) Rotation by π about the origin followed by reflection across the line passing through $(1, 1)$ and $(1, 2)$
    b) Rotation by $\pi/3$ about $(1, 1)$
    c) Reflection across the line passing through $(1, 1)$ and $(-1, -1)$
    d) Rotation by $\pi/2$ about $(1, 1)$ followed by reflection across the line $y = x$ followed by rotation by $-\pi/2$ about $(1, 1)$
    e) Reflection across the line $x = 1$ followed by reflection across the line $x = -1$

20. Which of the following maps $\mathbb{P}_2 \to \mathbb{R}^2$, using the following notation: $p(x) = a_2x^2 + a_1x + a_0$, is not a linear transformation?

    a) $p(x) \mapsto \begin{bmatrix}a_2\\a_1\end{bmatrix}$ b) $p(x) \mapsto \begin{bmatrix}p''(0)\\p'(2)\end{bmatrix}$ c) $p(x) \mapsto \begin{bmatrix}p(0)\\p(1)-1\end{bmatrix}$ d) $p(x) \mapsto \begin{bmatrix}p(1)\\a_0\end{bmatrix}$ e) $p(x) \mapsto \begin{bmatrix}p(9)-p(0)\\0\end{bmatrix}$

21. Which of the following maps $\mathbb{P}_n \to \mathbb{R}^n$ is an one-to-one linear transformation?

    a) $p(x) \mapsto \begin{bmatrix}p(1)\\p(2)\\\vdots\\p(n)\end{bmatrix}$ b) $p(x) \mapsto \begin{bmatrix}p'(1)\\p'(2)\\\vdots\\p'(n)\end{bmatrix}$ c) $p(x) \mapsto \begin{bmatrix}\int_0^1 p(t)dt\\\int_0^2 p(t)dt\\\vdots\\\int_0^n p(t)dt\end{bmatrix}$ d) $p(x) \mapsto \begin{bmatrix}p(1)\\p'(1)\\\vdots\\p^{(n)}(1)\end{bmatrix}$ e) None of them

22. Which of the following maps $\mathbb{P}_n \to \mathbb{P}_n$ is an onto linear transformation?

a) $p(x) \mapsto p'(x)$ b) $p(x) \mapsto p(2x)$ c) $p(x) \mapsto \int_0^{2x} p'(t)dt$ d) $p(x) \mapsto p'(x) + p(x) - p(0)$

23. Which of the following maps $\mathbb{P}_{n-1} \to \mathbb{R}^n$ is an isomorphism?

a) $p(x) \mapsto \begin{bmatrix} p(1) \\ p(2) \\ \vdots \\ p(n) \end{bmatrix}$ b) $p(x) \mapsto \begin{bmatrix} p'(1) \\ p'(2) \\ \vdots \\ p'(n) \end{bmatrix}$ c) $p(x) \mapsto \begin{bmatrix} \int_0^1 p(t)dt \\ \int_0^2 p(t)dt \\ \vdots \\ \int_0^n p(t)dt \end{bmatrix}$ d) $p(x) \mapsto \begin{bmatrix} p(1) \\ p'(1) \\ \vdots \\ p^{(n)}(1) \end{bmatrix}$

24. A subspace S of $\mathbb{R}^n$ is the **orthogonal complement** of a d-dimensional subspace T if and only if

a) Some vector in S is orthogonal to every vector in T
b) There are d linearly independent vectors in S orthogonal to every vector in T
c) Every vector in $\mathbb{R}^n$ can be expressed as the sum of two vectors each of which belongs to S and T
d) $\dim S = n - d$ and $S \cap T = \{\mathbf{0}\}$
e) S contains $(n-d)$ vectors orthogonal to every vector in T
f) $S \cap T = \{\mathbf{0}\}$ and $S \cup T = \mathbb{R}^n$

25. For which pair of real numbers (a, b) is the matrix $\begin{bmatrix} 1 & -2 & -1 \\ -1 & a & 1 \\ 3 & -6 & b \end{bmatrix}$ rank one?

a) $(-1, -3)$ b) $(2, 1)$ c) $(2, -3)$ d) $(-2, 3)$ e) None of them

26. What is the sum of the dimensions of the null space and column space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{bmatrix}?$$

a) 4 b) 5 c) 6 d) 7 e) 8

27. The rank of a matrix is*

a) # of rows
b) # of rows minus the dimension of the null space
c) # of columns
d) # of columns minus the dimension of the column space
e) # of columns minus the dimension of the null space

28. The following subspace in $\mathbb{R}^4$ is the orthogonal complement of

$$T := \{[x_1, x_2, x_3, x_4]^T \mid x_1 + x_2 + x_3 = 0,\ x_1 - x_2 - x_3 = 0\}:$$

a) Span$\{[2,0,0,0]^T,\ [0,1,-1,0]^T\}$
b) Span$\{[3,1,1,1]^T, [1,1,1,0]^T\}$
c) Span$\{[1,0,0,0]^T,\ [1,0,1,0]^T\}$
c) Span$\{[1,1,1,0]^T, [0,1,1,0]^T\}$
e) Span$\{[1,3,3,0]^T,\ [2,1,2,0]^T\}$

29. A least-squares solution to $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is

a) $\mathbf{x} = 1$ b) $\mathbf{x} = 2$ c) $\mathbf{x} = 3$ d) $\mathbf{x} = 4$

30. A least-squares solution to $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is

a) $\mathbf{x} = 1$ b) $\mathbf{x} = 2$ c) $\mathbf{x} = 3$ d) $\mathbf{x} = 4$

31. A least-squares solution $\hat{\mathbf{x}}$ of the system $A\mathbf{x} = \mathbf{b}$ is always characterized by the following:

a) $\hat{\mathbf{x}}$ is the shortest vector in Col$A$
b) $\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{b}$ onto Col$A$
c) $\mathbf{b}$ is the orthogonal projection of $\hat{\mathbf{x}}$ onto Col$A$
d) A$\hat{\mathbf{x}}$ is $\mathbf{b}$
e) A$\hat{\mathbf{x}}$ is the closest vector in Col$A$ to $\mathbf{b}$

32. Find the Fourier series of $\cos x$ on the interval $[-\pi, \pi]$
33. Find the Fourier series of $|\sin x|$ on the interval $[-\pi, \pi]$. What can you get for $x = 0$? or $x = \pi/2$? any other x's?
34. Find the Fourier series of $|\cos x|$ on the interval $[-\pi, \pi]$
35. Find the Fourier series of $\cos^2 x$ on the interval $[-\pi, \pi]$
36. Find the Fourier series of $\sin^2 x$ on the interval $[-\pi, \pi]$
37. Find the Fourier series of $|\sin x \cos x|$ on the interval $[-\pi, \pi]$
38. Find the Fourier cosine series of $\sin x$ on the interval $[0, \pi]$ and explain what you can get at $x = \pi/2$?
39. Describe the "Variation of Parameters" for solving the second-order inhomogeneous ODE

$$y''(t) + a_1(t)y'(t) + a_0(t)y(t) = g(t).$$

40. Find a solution to the initial value problem (WebWork #1)

$$y'(t) + \sin t \cdot y(t) = g(t), \quad y(0) = 7,$$

that is continuous on the interval $[0, 2\pi]$ where

$$g(t) = \begin{cases} \sin t & \text{if } 0 \le t \le \pi, \\ -\sin t & \text{if } \pi < t \le 2\pi \end{cases}.$$

41. Find the general solution to the differential equation (WebWork #2)

$$y' \cos x = y \sin x + \sin 175x \text{ for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

42. A 1-kg mass is attached to a spring with stiffness 17N/m. The damping constant for the system is 8N-sec/m. At some point, the mass was located 3m distance to the left of equilibrium and had velocity 17m/sec to right direction. What is the maximum displacement to the right that it will attain?
43. Given a linear second-order equation

$$y''(t) + ay'(t) + by(t) = f(t),$$

only information you have is a set of three solutions to the equation. They are

$$t + e^t \cos 2t + e^{2t} \sin t,\ t + e^{2t} \sin t,\ t + e^t \sin 2t + e^{2t} \sin t.$$

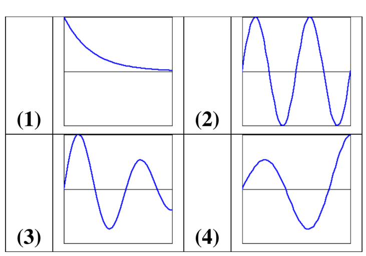
Find a, b, and $f(t)$.

44. The differential equation (WebWork #3)

$$x^2 \frac{d^2y}{dx^2} - 7x\frac{dy}{dx} + 16y = 0$$

has x^4 as a solution. Find another linearly independent solution.

45. Match the graphs of solutions shown in the figure below with each of the differential equations below (WebWork #5)
a) $x'' + 4x = 0$ b) $x'' - 4x = 0$ c) $x'' - 0.7x' + 1.1225x = 0$ d) $x'' + 0.7x' + 1.1225x = 0$



46. Find the solution to initial value problem (WebWork #11):

$$\frac{d^2y}{dt^2} - 14\frac{dy}{dt} + 49y = 0, \quad y(0) = 8, \quad y'(0) = 7.$$

47. Change the differential equation

$$y'''(t) + \frac{1}{1+t}y''(t) - y(t) = \sin t$$

into the form of

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$$

where $\mathbf{x}(t)$, $\mathbf{f}(t)$ are 3×1 matrices and $\mathbf{A}(t)$ is a 3×3 matrix.

48. Solve the following differential equation by variation of parameters (WebWork #17):

$$y'' + 16y = \sec 4x.$$

49. Let $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ be a solution to the system (WebWork #20)

$$\begin{array}{rcr} x_1'(t) & = & 8x_1(t) - 16x_2(t) \\ x_2'(t) & = & -8x_2(t) \end{array}.$$

If $\mathbf{x}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, find $\mathbf{x}(t)$.

50. Find the real-valued solution to the initial value problem $\mathbf{y}(t) = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ (WebWork#25):

$$\mathbf{y}(t)' = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \mathbf{y}(t).$$

51. Solve the system via eigenvector decomposition (WebWork #27):

$$\mathbf{x}(t)' = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \sin t \\ e^t \end{bmatrix}.$$

52. Decide if the following statements are *always true* or *sometimes false*.

a) Every orthogonal set is a linearly independent set.

b) Two diagonalizable matrices A and B are similar if they have the same eigenvalues, counting multiplicities.

c) If A^3 is diagonalizable, then A is diagonalizable as well.

d) If A^3 is diagonalizable, then there exists diagonalizable B such that $A^3 = B^3$.

e) Let A be a $n \times n$ matrix. If the sum of entries in a column is zero for each column, then 0 is an eigenvalue of A.

f) Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are vectors in $\mathbb{R}^n$. If $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is an orthonormal set, then it is a basis for $\mathbb{R}^n$.

g) If A and B are $n \times n$ invertible matrices, then AB is similar to BA.

h) Given a subspace W of V, the orthogonal projection map from V to W is a one-to-one linear transformation.

i) The orthogonal complement of the null space of A is the same as the column space of A if A is symmetric.

j) If the orthogonal complement of the null space of A is the same as the column space of A, then A is symmetric.

k) A square matrix A is invertible if and only if 0 is not an eigenvalue of A.

A mass-spring system

Given a mass-spring system

$$my'' + by' + ky = F_{ext}$$

m is the mass, b is the damping coefficient, and k is the stiffness.

Since we derived the equation from the real world, we implicitly assume that $m > 0$, $b \geq 0$, $k > 0$. When there is no friction force or $b = 0$, the system is called **undamped**. For $0 < b < \sqrt{4mk}$, the system is called **underdamped**. Meanwhile, the system is called overdamped when $b > \sqrt{4mk}$. As the last thing, when b is exactly $\sqrt{4mk}$, the system is said to be **critcally damped**. For the external force term, the system is called **free** if $F_{ext} = 0$.

For the sake of convenience, we define

$$\omega = \sqrt{\frac{k}{m}}$$

so that the equation becomes

$$y'' + \frac{b}{\sqrt{4mk}} 2\omega y' + \omega^2 y = \frac{F_{ext}}{m}$$

Undamped and Free The general solution is expressed in the form

$$A\cos(\omega t + \phi)$$

The period is $\frac{2\pi}{a}$ and the natural frequency is $\frac{a}{2\pi}$.

Underdamped and Free In this case, the general solution is expressed in the form

$$Ae^{\alpha t}\sin(\beta t + \phi)$$

The term $Ae^{\alpha t}$ is called an exponential **damping factor** and **quasiperiod** is $\frac{2\pi}{\beta}$ and **quasifrequency** is $\frac{\beta}{2\pi}$.

Overdamped and Free The general solution has the form as

$$c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Critically damped and Free The general solution has the form as

$$(c_1 + c_2 t)e^{rt}$$