Quiz 9

1. Consider the matrix A given by

$$\begin{bmatrix} 0 & -2 \\ 5 & -2 \end{bmatrix}.$$

Find a real matrix C of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

which is similar to A. In other words, find such a real matrix C and an invertible matrix P such that $A = PCP^{-1}$ (or equivalently, $P^{-1}AP = C$, or $AP = PC$).

The characteristic equation of A is

$$\det(A - \lambda I) = 0 \iff \lambda^2 + 2\lambda + 10 = 0.$$

Hence, we have $\lambda = -1 + 3i$ and $-1 - 3i$ as eigenvalues.

$$A - (-1 - 3i)I = \begin{bmatrix} 1 + 3i & -2 \\ 5 & -1 + 3i \end{bmatrix}.$$

Thus, $\left\{ \begin{bmatrix} 2 \\ 1 + 3i \end{bmatrix} \right\}$ is a basis for the eigenspace of $\lambda = -1 - 3i$. Now, let $P = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$. Then, we get

$$A = PCP^{-1} \text{ where } C = \begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix}.$$

2. Check if

$$\begin{bmatrix} 1/6 \\ 1/6 \\ 3/6 \\ 5/6 \end{bmatrix}, \begin{bmatrix} 3/6 \\ -5/6 \\ -1/6 \\ 1/6 \end{bmatrix}, \begin{bmatrix} 1/5 \\ 2/5 \\ 2/5 \\ 4/5 \end{bmatrix}$$

have length 1, respectively. Also, check if they are orthogonal to each other.

$1^2 + 1^2 + 3^2 + 5^2 = 1 + 1 + 9 + 25 = 36 = 6^2$, $3^2 + (-5)^2 + (-1)^2 + 1^2 = 6^2$, and $1^2 + 2^2 + 2^2 + 4^2 = 1 + 4 + 4 + 16 = 25$. Thus, they are all unit vectors (in other words, they have length 1). However, the dot product of 2nd and 3rd vectors gives $\frac{3-10-2+4}{30} \neq 0$ so that they are not orthogonal.

1. Consider the matrix A given by
$$\begin{bmatrix} 1 & -2 \\ 5 & -1 \end{bmatrix}.$$
Find a real matrix C of the form
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$
which is similar to A. In other words, find such a real matrix C and an invertible matrix P such that $A = PCP^{-1}$ (or equivalently, $P^{-1}AP = C$, or $AP = PC$).

The characteristic equation of A is
$$\det(A - \lambda I) = 0 \iff \lambda^2 + 9 = 0.$$
Hence, we have $\lambda = 3i$ and $-3i$ as eigenvalues.
$$A - 3iI = \begin{bmatrix} 1-3i & -2 \\ 5 & -1-3i \end{bmatrix}.$$
Thus, $\left\{ \begin{bmatrix} 2 \\ 1-3i \end{bmatrix} \right\}$ is a basis for the eigenspace of $\lambda = 3i$. Now, let $P = \begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}$. Then, we get
$$A = PCP^{-1} \text{ where } C = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}.$$

2. Check if
$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$
are orthogonal to each other.

You can check all total three dot products of 3 pairs of two vectors. They are all 0's. So, they are orthogonal to each other.