Quiz 8

Caution. You should NOT do row reductions for finding eigenVALUEs.

1. Is $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 0 & 7 & 2 \\ 6 & 4 & 6 \end{bmatrix}$?
   If so, what is the eigenvalue associated with the eigenvector?

$$\begin{bmatrix} 3 & 6 & 7 \\ 0 & 7 & 2 \\ 6 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \\ 10 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

Hence, $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 0 & 7 & 2 \\ 6 & 4 & 6 \end{bmatrix}$ with the eigenvalue 5.

2. Consider the matrix

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 0 & 1 & 0 \\ 2 & 3 & 5 \end{bmatrix}$$

1) Find all eigenvalues. **2)** Find a basis for each eigenspaces.

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1) The characteristic equation of A is $\det \begin{bmatrix} 3-\lambda & 3 & 4 \\ 0 & 1-\lambda & 0 \\ 2 & 3 & 5-\lambda \end{bmatrix} = 0$. Let's take the second row to use cofactor expansion. Then, the characteristic equation becomes

$$0 = (1-\lambda)((3-\lambda)(5-\lambda) - 8) = (1-\lambda)(1-\lambda)(7-\lambda).$$

So, eigenvalues are 1 and 7.

2) The eigenspaces are $\text{Nul}\,(A - I)$ and $\text{Nul}\,(A - 7I)$.

$$A - I = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 2 & 3 & 4 \end{bmatrix} \qquad \& \qquad A - 7I = \begin{bmatrix} -4 & 3 & 4 \\ 0 & -6 & 0 \\ 2 & 3 & -2 \end{bmatrix}$$

Hence, $E_1 = \text{Nul}\,(A - I) = \text{Span}\left\{ \begin{bmatrix} -1.5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $E_7 = \text{Nul}\,A - 7I = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.
Now, the sets come after the word Span are bases for each eigenspaces.

Caution. You should NOT do row reductions for finding eigenVALUEs.

1. Is $\begin{bmatrix}1\\0\\1\end{bmatrix}$ an eigenvector of $\begin{bmatrix}2 & 6 & 7\\3 & -1 & -3\\4 & 6 & 4\end{bmatrix}$?
   If so, what is the eigenvalue associated with the eigenvector?

$$\begin{bmatrix}2 & 6 & 7\\3 & -1 & -3\\4 & 6 & 4\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}9\\0\\8\end{bmatrix} \neq \lambda\begin{bmatrix}1\\0\\1\end{bmatrix}$$

for any $\lambda \in \mathbb{R}$. Hence, $\begin{bmatrix}1\\0\\1\end{bmatrix}$ is not an eigenvector of $\begin{bmatrix}2 & 6 & 7\\3 & -1 & -3\\4 & 6 & 4\end{bmatrix}$.

2. Consider the matrix

$$A = \begin{bmatrix}1 & 0 & 2\\1 & -1 & -1\\3 & 0 & 2\end{bmatrix}$$

1) Find all eigenvalues. **2)** Find a basis for each eigenspaces.[1]

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1) The characteristic equation of A is $\det\begin{bmatrix}1-\lambda & 0 & 2\\1 & -1-\lambda & -1\\3 & 0 & 2-\lambda\end{bmatrix} = 0$. Let's take the second column to use cofactor expansion. Then, the characteristic equation becomes

$$0 = (-1-\lambda)((1-\lambda)(2-\lambda)-6) = (-1-\lambda)(4-\lambda)(-1-\lambda).$$

So, eigenvalues are -1 and 4.

2) The eigenspaces are $\mathrm{Nul}\,(A-(-1)I)$ and $\mathrm{Nul}\,(A-4I)$.

$$A+I = \begin{bmatrix}2 & 0 & 2\\1 & 0 & -1\\3 & 0 & 3\end{bmatrix} \qquad \& \qquad A-4I = \begin{bmatrix}-3 & 0 & 2\\1 & -5 & -1\\3 & 0 & -2\end{bmatrix}$$

Hence, $E_{-1} = \mathrm{Nul}\ (A+I) = \mathrm{Span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ and $E_4 = \mathrm{Nul}\ A-4I = \mathrm{Span}\left\{\begin{bmatrix}10\\-1\\15\end{bmatrix}\right\}$.

Now, the sets come after the word Span are bases for each eigenspaces.

[1] This gives an example of a matrix and its eigevalue which has different algebraic multiplicity (2) from geometric multiplicty (1).