Quiz 6

Name : ______________________________

SID : ______________________________

1. Let a subset W of $M_{2\times 2}$ be given as

$$W = \left\{A \in M_{2\times 2} | A^2 = \mathbf{0}\right\}$$

where $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Is W a subspace of $M_{2\times 2}$? Why or why not?

Solution. Note that we have $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in W. However, their sum $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ becomes I_2 when squared. Hence, W is not a subspace of $M_{2\times 2}$.

2. Let $\mathcal{B} = \left\{ b_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, b_2 = \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}, b_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ be a basis for $\mathbb{R}^3$.

a) Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We are given that

$$\begin{aligned} e_1 &= 3b_1 - b_2 + 3b_3 \\ e_2 &= b_3 \\ e_3 &= 2b_1 - b_2 + 3b_3 \end{aligned}$$

Find the inverse matrix $P_{\mathcal{B}}^{-1}$ and explain why. (You can use the formula $v = P_{\mathcal{B}} \cdot [v]_{\mathcal{B}}$.)
Solution. Put $v = e_1$. Then, we get $P_{\mathcal{B}}^{-1} e_1 = [e_1]_{\mathcal{B}}$. Hence, $P_{\mathcal{B}}^{-1}$'s first column is $[e_1]_{\mathcal{B}}$.

$$P_{\mathcal{B}}^{-1} = \begin{pmatrix} 3 & 0 & 2 \\ -1 & 0 & -1 \\ 3 & 1 & 3 \end{pmatrix}.$$

b) Let $u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Find $[u]_{\mathcal{B}}$.

Solution. $[u]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} u = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix}$.

1. Let a subset W of $M_{2\times 3}$ be given as

$$W = \left\{ A \in M_{2\times 3} | \, A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Is W a subspace of $M_{2\times 3}$? Why or why not?

Solution 1. [Magic of linear transformation] Let's define a linear transformation $T : M_{2\times 3} \to M_{2\times 1}$ as following:

$$T(A) = A \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then, since $AC + BC = (A + B)C$ and $(cA)C = c(AC)$ for any matrices A, B, and C and a scalar c, so T is linear.

W is the kernel of T so that W is a subspace of $M_{2\times 3}$.

Solution 2. Since $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the zero vector $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is in W.

If $A, B \in W$, then $(A+B) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + B \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

You can also check for $cA \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. So, W is a subspace of $M_{2\times 3}$.

2. Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 7 \end{pmatrix} \right\}$ be a basis for $\mathbb{R}^3$.

   a) Explain why $P_{\mathcal{B}}$ is the matrix obtained by attaching basis vectors side by side. (Recall that the definition of $P_{\mathcal{B}}$ is the matrix satisfying

   $$v = P_{\mathcal{B}} \cdot [v]_{\mathcal{B}}$$

   for all $v \in \mathbb{R}^3$.)

   Solution. $P_{\mathcal{B}}$'s first column is $P_{\mathcal{B}} e_1 = P_{\mathcal{B}}[b_1]_{\mathcal{B}} = b_1$. So do other columns. So, $P_{\mathcal{B}}$ is the matrix obtained by attaching basis vectors side by side.

   b) Find $v \in \mathbb{R}^3$ such that $[v]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

   Solution. $v = P_{\mathcal{B}} \cdot [v]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$.