Name (Last, First): ______________________________

Student ID: ______________________________

1. Determine if the columns of the matrix form a linearly independent set.

$$\begin{bmatrix} 0 & 2 & 3 \\ 1 & 3 & 6 \\ -1 & 1 & 0 \end{bmatrix}$$

Solution 1. **The set is linearly dependent.** This is because

$$3\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so that the three columns form a linearly dependent set.

Solution 2. Let's take the associated augmented matrix for the homogeneous equation.

$$\begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 3 & 6 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

In order to make the matrix to be in Row Echelon Form, we need the below row reduction steps.

$$\begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 3 & 6 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 6 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 6 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 6 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first $\sim$ is obtained by interchanging 1st and 2nd rows.
The second $\sim$: Changing 3rd row into 1st row + 3rd row.
The last $\sim$: Changing 3rd row into 3rd row + $(-2)\times$ 2nd row.

Hence, we can find a solution $z =$free from 3rd row. $y = -\frac{3}{2}z$ from 2nd row. $x = y$ from 1st and 2nd row. Find one example of x, y, and z by setting $z = -2$, we get the weights x, y, and z (not all zeros) that make the linear combination $x\mathbf{x}_1 + y\mathbf{x}_2 + z\mathbf{x}_3$ become zero.

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2. Let $T(x, y) = (2x + y, x)$. Show that T is a **one-to-one** linear transformation. Does T map $\mathbb{R}^2$ **onto** $\mathbb{R}^2$?

Solution. First of all,

$$\begin{aligned} T((x_1, y_1)) + T((x_2, y_2)) &= (2x_1 + y_1, x_1) + (2x_2 + y_2, x_2) \\ &= (2(x_1 + x_2) + (y_1 + y_2), x_1 + x_2) = T((x_1 + x_2, y_1 + y_2)) \end{aligned}$$

Also, $T(c(x, y)) = (2cx + cy, cx) = c(2x + y, x) = cT((x, y))$. Hence, T is a linear transformation.

Now, in order to prove that T is one-to-one, (because T is a linear transformation and by Theorem 11 (Chapter 1.9)) we only need to show that

If $T(x, y) = (0, 0)$ then $x = 0$ and $y = 0$.

Suppose that $T(x, y) = (0, 0)$, then it implies that $2x + y = 0$, $x = 0$. So, obviously, you get $x = 0$ and $y = 0$. Henceforth, T is a one-to-one linear transformation.

For the last question, the answer is **YES**. To get this answer, we need the argument below.

For an arbitrary element in $\mathbb{R}^2$, say (z, w), if we define $x = w$, $y = z - 2w$ then $T(x, y) = (2x + y, x) = (2w + z - 2w, w) = (z, w)$.

Therefore, T maps $\mathbb{R}^2$ onto $\mathbb{R}^2$.