Quiz 13

1. Find a general solution to the homogeneous equation:

$$\left(\frac{d}{dt} - 2\right)^2 \left(\frac{d^2}{dt^2} - 2\frac{d}{dt} + 2\right)^2 y = 0$$

The auxiliary equation is

$$(r-2)^2(r^2-2r+2)^2 = 0.$$

Since 2 repeats twice, we have e^{2t} and te^{2t}. Similarly, $1 \pm i$ repeat twice each, so we have solutions $e^t \cos t$, $e^t \sin t$, and $te^t \cos t$, $te^t \sin t$. Since it is 6-th order linear differential equation, the solution set has dimension 6 and the solutions we have found by far are linearly independent 6 solutions. So, a general solution is a linear combination of them.

Answer. $y(t) = c_1 e^{2t} + c_2 te^{2t} + c_3 e^t \cos t + c_4 e^t \sin t + c_5 te^t \cos t + c_6 e^t \sin t$

2. Let

$$\mathbf{x}_1 = \begin{bmatrix} 2e^{4t} \\ e^{4t} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}.$$

Determine if $\{\mathbf{x}_1, \mathbf{x}_2\}$ form a fundamental solution set of the system:

$$\mathbf{x}' = \begin{bmatrix} 5 & -2 \\ 3 & -2 \end{bmatrix} \mathbf{x}.$$

Note that

$$\begin{bmatrix} 5 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 5 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = - \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

1) $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions for the system:

$$\mathbf{x}_1' = \begin{bmatrix} 8e^{4t} \\ 4e^{4t} \end{bmatrix} = e^{4t} \cdot 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = e^{4t} \cdot \begin{bmatrix} 5 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 3 & -2 \end{bmatrix} \mathbf{x}_1.$$

$$\mathbf{x}_2' = \begin{bmatrix} -e^{-t} \\ -3e^{-t} \end{bmatrix} = e^{-t} \cdot - \begin{bmatrix} 1 \\ 3 \end{bmatrix} = e^{-t} \cdot \begin{bmatrix} 5 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 3 & -2 \end{bmatrix} \mathbf{x}_2.$$

2) $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent:

The Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$ is

$$\det \begin{bmatrix} 2e^{4t} & e^{-t} \\ e^{4t} & 3e^{-t} \end{bmatrix} = 5e^{3t} \not\equiv 0.$$

So, they are linearly independent. **As a result,** $\{\mathbf{x}_1, \mathbf{x}_2\}$ form a fundamental solution set.

1. Find a general solution to the homogeneous equation:

$$\left(\frac{d}{dt}-1\right)^2\left(\frac{d^2}{dt^2}-\frac{d}{dt}\right)y=0$$

The auxiliary equation is $(r-1)^2(r^2-r)=0$, so $r=1$ appears three times and $r=0$ appears once. So,

Answer. $y(t)=c_1+c_2e^t+c_3te^t+c_4t^2e^t$.

2. Let

$$\mathbf{x}_1=\begin{bmatrix}e^{-t}\\0\\e^{-t}\end{bmatrix},\quad \mathbf{x}_2=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad \mathbf{x}_3=\begin{bmatrix}e^{3t}\\0\\-e^{3t}\end{bmatrix}.$$

Determine if $\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\}$ form a fundamental solution set of the system:

$$\mathbf{x}'=\begin{bmatrix}1&0&-2\\0&0&0\\-2&0&1\end{bmatrix}\mathbf{x}.$$

Note that

$$\begin{bmatrix}1&0&-2\\0&0&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}=-\begin{bmatrix}1\\0\\1\end{bmatrix},\quad \begin{bmatrix}1&0&-2\\0&0&0\\-2&0&1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=0\begin{bmatrix}0\\1\\0\end{bmatrix},\quad \begin{bmatrix}1&0&-2\\0&0&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}=3\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

1) $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are solutions for the system:

$$\mathbf{x}_1'=\begin{bmatrix}-e^{-t}\\0\\-e^{-t}\end{bmatrix}=e^{-t}\cdot-\begin{bmatrix}1\\0\\1\end{bmatrix}=e^{-t}\cdot\begin{bmatrix}1&0&-2\\0&0&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}1&0&-2\\0&0&0\\-2&0&1\end{bmatrix}\mathbf{x}_1.$$

$$\mathbf{x}_2'=\begin{bmatrix}0\\0\\0\end{bmatrix}=0\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1&0&-2\\0&0&0\\-2&0&1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1&0&-2\\0&0&0\\-2&0&1\end{bmatrix}\mathbf{x}_1.$$

$$\mathbf{x}_3'=\begin{bmatrix}3e^{3t}\\0\\-3e^{3t}\end{bmatrix}=e^{3t}\cdot 3\begin{bmatrix}1\\0\\-1\end{bmatrix}=e^{3t}\cdot\begin{bmatrix}1&0&-2\\0&0&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}1&0&-2\\0&0&0\\-2&0&1\end{bmatrix}\mathbf{x}_1.$$

2) $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are linearly independent:

The Wronskian $W[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3](t)$ is

$$\det\begin{bmatrix}e^{-t}&0&e^{3t}\\0&1&0\\e^{-t}&0&e^{-3t}\end{bmatrix}=1\cdot\det\begin{bmatrix}e^{-t}&e^{3t}\\e^{-t}&-e^{3t}\end{bmatrix}=-2e^{2t}\not\equiv 0.$$

So, they are linearly independent. **As a result,** $\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\}$ form a fundamental solution set.