Quiz 12

1. Solve the initial value problem

$$y'' - 5y' + 6y = -e^{2t}, \quad y(0) = 2, \quad y'(0) = 5.$$

First of all, let's find a particular solution.

- The auxiliary equation : $r^2 - 5r + 6 = 0$ or $(r-2)(r-3) = 0$.
- 2 that appears at $2t$ in $-e^{2t}$ becomes a solution to the auxiliary equation.
- So, let's try Ate^{2t}, not just Ae^{2t}.

Let $y_p(t) = Ate^{2t}$, then $y_p'(t) = A(1+2t)e^{2t}$ and $y_p''(t) = A(4+4t)e^{2t}$. So, $y_p''(t) - 5y_p'(t) + 6y_p(t) = -Ae^{2t}$. Hence, $A = 1$. Now, a general solution to the differential equation (if we assume that our answer does not necessarily satisfy the initial condition) will be

$$y(t) = te^{2t} + c_1 e^{2t} + c_2 e^{3t}.$$

Now, using the initial values given, let's solve for c_1 and c_2.

- $y(0) = c_1 + c_2$ which is supposed to be 2.
- Since $y'(t) = (1+2t)e^{2t} + 2c_1 e^{2t} + 3c_2 e^{3t}$, $y'(0) = 1 + 2c_1 + 3c_2$ and it should be 5.
- Solving those two linear equations, we get $c_1 = 2$ and $c_2 = 0$.

Answer. $y(t) = 2e^{2t} + te^{2t}$ (or $(t+2)e^{2t}$)

2. Find a general solution to the following differential equation.

$$y'' + 4y = 8e^{2t} + 4\cos 2t$$

- The auxiliary equation is $r^2 + 4 = 0$ or $(r+2i)(r-2i) = 0$.
- $\cos 2t$ is $\frac{1}{2}(e^{2it} + e^{-2it})$ and $2i$ and $-2i$ are solutions for the auxiliary equation.
- We try $At\cos 2t + Bt\sin 2t$ for $+4\cos 2t$ part.

Let's first do $4\cos 2t$ part. Our guess is $y_p(t) = At\cos 2t + Bt\sin 2t$. Then,

$$y_p'(t) = A\cos 2t + B\sin 2t - 2At\sin 2t + 2Bt\cos 2t$$

and $y_p''(t) = -2A\sin 2t + 2B\cos 2t - 2A\sin 2t + 2B\cos 2t - 4At\cos 2t - 4Bt\sin 2t$. So, $y'' + 4y = -4A\sin 2t + 4B\cos 2t$ and we get $A = 0$, $B = 1$. So, $y_p(t) = t\sin 2t$ works for $4\cos 2t$ on the RHS.

For $8e^{2t}$, since 2 is not a root for $r^2 + 4 = 0$ (note that it is "+" not "$-$"), let $y_p(t) = Ae^{2t}$ this time. Then, $y_p'(t) = 2Ae^{2t}$ and $y_p''(t) = 4Ae^{2t}$. Hence, $y'' + y = 8Ae^{2t}$, so $A = 1$.

Now, we have a particular solution for the equation : $e^{2t} + t\sin 2t$. Now, a general solution will be

Answer. $y(t) = e^{2t} + t\sin 2t + c_1\cos 2t + c_2\sin 2t$

1. Solve the initial value problem

$$y'' + 3y' + 2y = -e^{-2t}, \quad y(0) = 2, \quad y'(0) = -2.$$

First of all, let's find a particular solution.

- The auxiliary equation : $r^2 + 3r + 2 = 0$ or $(r+1)(r+2) = 0$.
- -2 that appears at $-2t$ in $-e^{-2t}$ becomes a solution to the auxiliary equation.
- So, let's try Ate^{-2t}, not just Ae^{-2t}.

Let $y_p(t) = Ate^{-2t}$, then $y_p'(t) = A(1-2t)e^{-2t}$ and $y_p''(t) = A(-4+4t)e^{-2t}$. So, $y_p''(t) + 3y_p'(t) + 2y_p(t) = -Ae^{-2t}$. Hence, $A = 1$. Now, a general solution to the differential equation (if we assume that our answer does not necessarily satisfy the initial condition) will be

$$y(t) = te^{-2t} + c_1e^{-1t} + c_2e^{-2t}.$$

Now, using the initial values given, let's solve for c_1 and c_2.

- $y(0) = c_1 + c_2$ which is supposed to be 2.
- Since $y'(t) = (1-2t)e^{-2t} - c_1e^{-t} - 2c_2e^{-2t}$, $y'(0) = 1 - c_1 - 2c_2$ and it should be -2.
- Solving those two linear equations, we get $c_1 = 1$ and $c_2 = 1$.

Answer. $y(t) = e^{-t} + e^{-2t} + te^{-2t}$ (or $e^{-t} + (t+1)e^{-2t}$)

2. Find a general solution to the following differential equation.

$$\frac{d^2y}{dt^2} + y = 2e^t - 2\sin t$$

- The auxiliary equation is $r^2 + 1 = 0$ or $(r+i)(r-i) = 0$.
- $\sin t$ is $\frac{1}{2i}(e^{it} - e^{-it})$ and i and $-i$ are solutions for the auxiliary equation.
- We try $At\cos t + Bt\sin t$ for $-2\sin t$ part.

Let's first do $-2\sin t$ part. Our guess is $y_p(t) = At\cos t + Bt\sin t$. Then,

$$y_p'(t) = A\cos t + B\sin t - At\sin t + Bt\cos t$$

and $y_p''(t) = -A\sin t + B\cos t - A\sin t + B\cos t - At\cos t - Bt\sin t$. So, $y'' + y = -2A\sin t + 2B\cos t$ and we get $A = 1$, $B = 0$. So, $y_p(t) = t\cos t$ works for $-2\sin t$ on the RHS.

For $2e^t$, since 1 is not a root for $r^2 + 1 = 0$ (note that it is "$+$" not "$-$"), let $y_p(t) = Ae^t$ this time. Then, $y_p'(t) = Ae^t$ and $y_p''(t) = Ae^t$. Hence, $y'' + y = 2Ae^t$, so $A = 1$.

Now, we have a particular solution for the equation : $e^t + t\cos t$. Now, a general solution will be

Answer. $y(t) = e^t + t\cos t + c_1\cos t + c_2\sin t$