Quiz 11

1. Solve the initial value problem

$$\frac{d^2y}{dt^2} + y = 0, \quad y(0) = 1,\ y'(0) = 1.$$

The auxiliary equation is $r^2 + 1 = 0$, so it has complex roots $r = \pm i$. So, $\{\sin t, \cos t\}$ spans the solution set. Let $y(t) = c_1 \sin t + c_2 \cos t$. Then, $y(0) = c_2$ and $y'(0) = c_1$, so we get $c_1 = c_2 = 1$.

Answer. $y(t) = \cos t + \sin t$.

2. Do *orthogonally diagonalization* for

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$A - \lambda I = \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix}$. So, the characteristic polynomial is $-\lambda(\lambda^2 - 1)$. There are 3 eigenvalues : -1, 0, 1. Respectively, eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. For **orthogonally** diagonalization, we need an orthonormal set. So, let

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, they satisfy $A = PDP^{-1}$, P is orthogonal, and D is diagonal.

1. Do *orthogonally diagonalization* for
$$A = \begin{bmatrix} 0 & 3 \\ 3 & 8 \end{bmatrix}$$
[Hint] *Orthogonally diagonalize* = Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$ (or $A = PDP^{-1}$).

$A - \lambda I = \begin{bmatrix} -\lambda & 3 \\ 3 & 8-\lambda \end{bmatrix}$. So, the characteristic polynomial is $\lambda^2 - 8\lambda - 9 = (\lambda - 9)(\lambda + 1)$. There are 2 eigenvalues : -1 and 9. Corresponding eigenvectors are $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Since we need an orthonormal set other than just an orthogonal set, we have to normalize those vectors and let
$$P = \begin{bmatrix} \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix} \text{ and } D = \begin{bmatrix} -1 & 0 \\ 0 & 9 \end{bmatrix}.$$
Then, P is an orthogonal matrix and D is diagonal. Also, they satisfy $A = PDP^{-1}$ (or $A = PDP^T$).

2. Solve the initial value problem
$$y'' - 5y' - 6y = 0, \quad y(0) = 1,\ y'(0) = -1.$$

The auxiliary equation is $r^2 - 5r - 6 = 0$ and $(r-6)(r+1) = 0$. So, a general solution is of the form
$$y(t) = c_1 e^{6t} + c_2 e^{-t}.$$
On one hand, $y(0) = c_1 + c_2$. On the other hand, $y(0) = 1$ is given. Similarly, $y'(0) = -1$ and $y'(0) = 6c_1 - c_2$. Hence, $c_1 = 0$, $c_2 = 1$.

Answer. $y(t) = e^{-t}$.