Quiz 10

1. Let W be the subspace of $\mathbb{R}^3$ spanned by two vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}3\\-1\\5\end{bmatrix}$. Find the vector in W which is closest (among vectors in W) to $\begin{bmatrix}-1\\-1\\6\end{bmatrix}$.

Since every vector in W is of the form $A\mathbf{x}$ where $A = \begin{bmatrix}1 & 3\\1 & -1\\1 & 5\end{bmatrix}$, this problem asks to find $A\mathbf{x}$ such that $||A\mathbf{x} - \mathbf{b}||$ is minimum where $\mathbf{b} = \begin{bmatrix}-1\\-1\\6\end{bmatrix}$. Here, we can use *normal equation* to find it. Note that

$$A^T A = \begin{bmatrix}3 & 7\\7 & 35\end{bmatrix} \text{ and } A^T\mathbf{b} = \begin{bmatrix}4\\28\end{bmatrix}.$$

According to the theory, we only need to find $\mathbf{x}$ such that $A^T A\mathbf{x} = A^T\mathbf{b}$. Solving this equation, we get $\mathbf{x} = \begin{bmatrix}-1\\1\end{bmatrix}$. This is not the answer, though. The answer is $w = A\mathbf{x}$.

Answer. $\begin{bmatrix}2\\-2\\4\end{bmatrix}$

2. Let H be the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}6\\1\\-2\\5\end{bmatrix}, \begin{bmatrix}0\\8\\2\\1\end{bmatrix}$. Find an orthogonal basis for H.

Apply Gram-Schmidt process. Just be careful that everytime you compute $v_n - \frac{v_n \cdot u_1}{u_1 \cdot u_1} u_1 - \cdots$, you HAVE TO use an ORTHOGONAL set $\{u_1, \cdots, u_{n-1}\}$.

Answer. $\left\{ \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}3\\1\\-5\\2\end{bmatrix}, \begin{bmatrix}-1\\8\\1\\0\end{bmatrix} \right\}$

1. Let W be given by

$$\text{Span}\left\{\begin{bmatrix}1\\2\\3\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\\-1\end{bmatrix}\right\}.$$

Find the vector $\mathbf{w}$ in W which is (among vectors in W) closest to $\begin{bmatrix}0\\3\\2\\0\end{bmatrix}$.

[Hint] You might use *normal equation* method. This is essentially the same question as the question about minimizing $||A\mathbf{x} - \mathbf{b}||$. WHY?

Since every vector in W is of the form $A\mathbf{x}$ where $A = \begin{bmatrix}1 & -1\\2 & 1\\3 & -1\\1 & -1\end{bmatrix}$, this problem asks to find $A\mathbf{x}$ such that $||A\mathbf{x} - \mathbf{b}||$ is minimum where $\mathbf{b} = \begin{bmatrix}0\\3\\2\\0\end{bmatrix}$. Here, we can use *normal equation* to find it. Note that

$$A^TA = \begin{bmatrix}15 & -3\\-3 & 4\end{bmatrix} \text{ and } A^T\mathbf{b} = \begin{bmatrix}12\\1\end{bmatrix}.$$

Solving this, we get $\mathbf{x} = \begin{bmatrix}1\\1\end{bmatrix}$. Note that this is NOT the correct answer, though. The answer is $w = A\mathbf{x}$.

Answer. $\begin{bmatrix}0\\3\\2\\0\end{bmatrix}$[1]

2. Let H be the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix}1\\-1\\2\\-1\end{bmatrix}, \begin{bmatrix}4\\1\\6\\1\end{bmatrix}, \begin{bmatrix}3\\3\\4\\1\end{bmatrix}$. Find an orthogonal basis for H.

Apply Gram-Schmidt process. Just be careful that everytime you compute $v_n - \frac{v_n \cdot u_1}{u_1 \cdot u_1}u_1 - \cdots$, you HAVE TO use an ORTHOGONAL set $\{u_1, \cdots, u_{n-1}\}$.

Answer. $\left\{\begin{bmatrix}1\\-1\\2\\-1\end{bmatrix}, \begin{bmatrix}2\\3\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}\right\}$.

[1] Actually, in this case $\mathbf{b}$ is already in W.