Practice Midterm 2

Student ID : ______________________________

Name : ______________________________

Problem	Score
1	/♡
2	/♡
3	/♡
4	/♡
5	/♡
6	/♡
Total	/6♡

Problem 1

Decide if the following statements are *always true* or *sometimes false*. JUSTIFY YOUR ANSWER.

a) Every orthogonal set is a linearly independent set.

b) Two diagonalizable matrices A and B are similar if they have the same eigenvalues, counting multiplicities.

c) If A^3 is diagonalizable, then A is diagonalizable as well.

d) If A^3 is diagonalizable, then there exists diagonalizable B such that $A^3 = B^3$.

e) Let A be a $n \times n$ matrix. If the sum of entries in a column is zero for each column, then 0 is an eigenvalue of A.

f) Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ are vectors in $\mathbb{R}^n$. If $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is an orthonormal set, then it is a basis for $\mathbb{R}^n$.

g) If A and B are $n \times n$ invertible matrices, then AB is similar to BA.

Problem 2

Define a linear transformation T from $\mathbb{P}_2$ to $\mathbb{P}_2$ as follows.

$$T(p(t)) = 3p(t) - tp'(t).$$

a) Let $\mathcal{E}$ be the standard basis for $\mathbb{P}_2$. Find the $\mathcal{E}$-matrix for T.

b) Is it possible to find a basis $\mathcal{B}$ for $\mathbb{P}_2$ such that

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}?$$

Problem 3

Let A be

$$\begin{bmatrix} 3 & -4 & -4 \\ 2 & 1 & -4 \\ -2 & 0 & 5 \end{bmatrix}$$

whose characteristic polynomial $\chi_A(\lambda)$ is $-(\lambda-1)(\lambda-3)(\lambda-5)$.

a) Find 3 linearly independent eigenvectors and, using them, find a diagonal matrix D and an invertible matrix P such that

$$P^{-1}AP = D.$$

b) Find all possible D's. For each D, find one corresponding invertible matrix P such that $P^{-1}AP = D$.

Problem 4

1) Let T be a linear transformation from V to W. For bases $\mathcal{B}$ of V and $\mathcal{C}$ of W, let the matrix for T relative to $\mathcal{B}$ and $\mathcal{C}$ be

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}.$$

Which of the following matrices could be a matrix for T (possibly, choosing different $\mathcal{B}'$ and $\mathcal{C}'$ from $\mathcal{B}$ and $\mathcal{C}$)?

a) $\begin{bmatrix}1&-1&2\\1&0&1\\0&0&0\end{bmatrix}$ b) $\begin{bmatrix}0&0&1\\0&0&0\\1&1&1\end{bmatrix}$ c) $\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}$ d) $\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}$ e) $\begin{bmatrix}0&-1&0\\-1&0&0\\1&1&0\end{bmatrix}$

2) Which of the following matrices are similar to

$$\begin{bmatrix}1&1&0\\0&1&0\\0&0&2\end{bmatrix}?$$

a) $\begin{bmatrix}1&2&3\\0&3&1\\0&0&2\end{bmatrix}$ b) $\begin{bmatrix}1&-1&2\\0&-1&0\\-1&1&1\end{bmatrix}$ c) $\begin{bmatrix}1&2&0\\0&1&0\\0&0&2\end{bmatrix}$ d) $\begin{bmatrix}1&0&3\\0&1&-1\\0&0&2\end{bmatrix}$ e) $\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}$

3) Which of the following sets are orthogonal?

a) $\left\{\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\-1\end{bmatrix},\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}$ b) $\left\{\begin{bmatrix}5\\2\end{bmatrix},\begin{bmatrix}-2\\5\end{bmatrix}\right\}$ c) $\left\{\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}2\\-2\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$

d) $\left\{\begin{bmatrix}1\\0\\3\\0\end{bmatrix},\begin{bmatrix}-3\\2\\1\\4\end{bmatrix},\begin{bmatrix}1\\-1\\1\\1\end{bmatrix}\right\}$ e) $\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\begin{bmatrix}1\\0\\0\\-1\end{bmatrix},\begin{bmatrix}2\\-3\\-1\\2\end{bmatrix},\begin{bmatrix}2\\1\\5\\2\end{bmatrix},\begin{bmatrix}6\\-1\\-11\\6\end{bmatrix}\right\}$

Problem 5

Consider

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

Note that they are orthogonal to each other and let W be the span of $\{\mathbf{u}, \mathbf{v}\}$.

a) Define a linear transformation T from $\mathbb{R}^4$ to $\mathbb{R}^4$ as the orthogonal projection

$$T(\mathbf{x}) = \text{proj}_W(\mathbf{x}) = \frac{\mathbf{u} \cdot \mathbf{x}}{3}\mathbf{u} + \frac{\mathbf{v} \cdot \mathbf{x}}{3}\mathbf{v}.$$

Let's denote the $\mathcal{E}$-matrix of T by $[T]$. ($\mathcal{E}$ is the standard basis for $\mathbb{R}^4$.) Find eigenvalues of $[T]$.

b) Is the matrix $[T]$ diagonalizable?

Problem 6[1]

Let W be a subspace of $\mathbb{R}^n$. Given an orthogonal basis $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_m\}$ for W, recall that the formula of the orthogonal projection of $v \in \mathbb{R}^n$ onto W is given by

$$\frac{\mathbf{b}_1 \cdot v}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 + \cdots + \frac{\mathbf{b}_m \cdot v}{\mathbf{b}_m \cdot \mathbf{b}_m}\mathbf{b}_m.$$

Let's denote this by $\text{proj}_{W,\mathcal{B}}(v)$.[2]

a) Show that $v - \text{proj}_{W,\mathcal{B}}(v)$ is orthogonal to $\text{proj}_{W,\mathcal{B}}(v)$. Also, prove that $v - \text{proj}_{W,\mathcal{B}}(v) \in W^{\perp}$.[3]

[1] This problem is designed to prove that the formula for the orthogonal projection,

$$\frac{\mathbf{b}_1 \cdot v}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 + \cdots + \frac{\mathbf{b}_m \cdot v}{\mathbf{b}_m \cdot \mathbf{b}_m}\mathbf{b}_m,$$

is independent of the choice of an orthogonal basis $\{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_m\}$ for W.

[2] I intentionally put $\mathcal{B}$ to emphasize that this is the projection using the basis $\mathcal{B}$.

[3] Hint. Use the linearity property of an innder product $\cdots$ and the definition of *orthogonality*. In order to prove $v - \text{proj}_{W,\mathcal{B}} \in W^{\perp}$, you only need to show that $v - \text{proj}_{W,\mathcal{B}}$ is orthogonal to $\mathbf{b}_1$, $\mathbf{b}_2$, $\cdots$, $\mathbf{b}_m$.

b) Let $\mathcal{C} = \{\mathbf{c}_1, \cdots, \mathbf{c}_m\}$ be another orthogonal basis for W.[4] Prove that[5]

$$\text{proj}_{W,\mathcal{B}}(v) - \text{proj}_{W,\mathcal{C}}(v) \in W^{\perp}.$$

c) Assume that there is no nonzero vector v such that $v \in W$ and $v \in W^{\perp}$ at the same time, without a proof. Using this fact, prove that

$$\text{proj}_{W,\mathcal{B}}(v) - \text{proj}_{W,\mathcal{C}}(v) = 0$$

Therefore,

$$\text{proj}_{W,\mathcal{B}}(v) = \text{proj}_{W,\mathcal{C}}(v).$$

So, we can conclude that the formula of the orthogonal projection does not depend on the choice of an orthogonal basis.

Remark. Why does $v \in W$ and $v \in W^{\perp}$ at the same time imply $v = 0$?

If then, $v \cdot v = 0$ because $v \in W$ and $v \in W^{\perp}$. However, $||v||^2 = 0$ implies $v = 0$.

[4] From a), we have $v - \text{proj}_{W,\mathcal{C}} \in W^{\perp}$.

[5] Hint. $W^{\perp}$ is a subspace of $\mathbb{R}^n$ (you can use this fact without a proof) so that $W^{\perp}$ is closed under addition and scalar multiplication.