Solution 8

1. Evaluate the double integral.

$$\iint_R (1+3x^2)dA, \quad R = \{(x,y) : -1 \le x \le 1, -1 \le y \le 1\}$$

Solution.
$$\int_{-1}^{1}\int_{-1}^{1}(1+3x^2)dxdy = (1-(-1)) \times \int_{-1}^{1}(1+3x^2)dx$$
since $(1+3x^2)$ is not a function containing y-term. Then,
$$2\int_{-1}^{1}(1+3x^2)dx = 2\left(x+x^3\right)\Big|_{-1}^{1} = 8.$$

Answer. 8

$$\iint_R (1+4x^3)dA, \quad R = \{(x,y) : -1 \le x \le 1, 0 \le y \le 2\}$$

Solution.
$$\int_{0}^{2}\int_{-1}^{1}(1+4x^3)dxdy = 2 \times \int_{-1}^{1}(1+4x^3)dx$$
since $(1+4x^3)$ is not a function containing y-term. Then,
$$2\int_{-1}^{1}(1+4x^3)dx = 2\left(x+x^4\right)\Big|_{-1}^{1} = 4.$$

Answer. 4

2. Use Lagrange multipliers to find the maximum and minimum values of the functions subject to the given constraint(s)

   a) $f(x,y,z) = x^3+y^3+z^3$; $x^2+y^2+z^2 = 3$ or 12

   Solution. By Lagrange Multipliers method, setting $g(x,y,z) = x^2+y^2+z^2-3$ or 12, we know that the maximum and minimum, if exist, is attained at P such that $\nabla f(P) \parallel \nabla g(P)$ ($\parallel$ means parallel), that is, $3(x^2,y^2,z^2) \parallel 2(x,y,z)$. So, we can say that $(x^2,y^2,z^2) \parallel (x,y,z)$ since multiplication by scalar just scales a vector but does not change the direction. Now, $(x^2,y^2,z^2) = \lambda(x,y,z)$ for some $\lambda \neq 0$.

   So, we have three equations $x^2 = \lambda x$, $y^2 = \lambda y$, $z^2 = \lambda z$. Hence, there are 2 possible cases for each x, y, z that they are one of 0 or λ.
   1) If they are nonzero, then $x^2+y^2+z^2 = 3$ gives $3\lambda^2 = 3$ or 12 so that $\lambda = \pm 1$ or ± 2. In this case, $f(x,y,z) = \pm 3$ or ± 24
   2) If exactly one of them is zero, we get $2\lambda^2 = 3$ or 12. In this case, $f(x,y,z) = \pm 3\sqrt{\frac{27}{8}}$ or $\pm 36\sqrt{6}$.
   3) If exactly one of them is nonzero, we get $\lambda^2 = 3$ or 12 and $f(x,y,z) = \pm 3\sqrt{3}$ or $\pm 24\sqrt{3}$.
   Lastly, note that $3\sqrt{3} > 3\sqrt{\frac{27}{8}} > 3$ or $24\sqrt{3} > 36\sqrt{6} > 24$.

   Answer. Max : $3\sqrt{3}$ or $24\sqrt{3}$ for each problems, Min : $-3\sqrt{3}$ or $-24\sqrt{3}$.

b) $f(x,y,z) = x^2 + y^2 + z^2$; $x + y = 3$ or 2, $2x + 3y + 2z = 3$ or 5
Solution. Just recall that the condition for Lagrange Multipliers method (for two constraints) is

$$\nabla f \in \operatorname{span}\{\nabla g, \nabla h\}.$$

And as we have discussed in section class we can write down that condition in a different way ;

$$(\nabla g \times \nabla h) \cdot \nabla f = 0.$$

For this problem, $\nabla g = (1,1,0)$ and $\nabla h = (2,3,2)$ so that $\nabla g \times \nabla h = (2,-2,1)$. Since $\nabla f = 2(x,y,z)$, the condition will be $2(2x - 2y + z) = 0$, so $2x - 2y + z = 0$. Now, we only need to solve three linear equations.

$$\begin{aligned} x + y &= 3 \text{ or } 2 \\ 2x + 3y + 2z &= 3 \text{ or } 5 \\ 2x - 2y + z &= 0 \end{aligned}$$

They have a solution $(x,y,z) = (2,1,-2)$ or $(1,1,0)$. Since we have only one point satisfying Lagrange Multipliers method condition, there would be no maximum or no minimum. To check this, it is enough to compute f for other point that is different from $(2,1,-2)$ or $(1,1,0)$.

First find a point lying on $x + y = 3$ and $2x + 3y + 2z = 3$, for example $(0,3,-3)$ and then calculate $f(0,3,-3) = 18 > 9 = f(2,1,-2)$. Hence, $f(2,1,-2) = 9$ is the minimum.

Answer. Max : Does not exist, Min : 9 or 2.

Letter grade for Quiz 8

$A^+ = 30$	(5)
$A0 = 26$	(1)
$A^- = 25$	(4)
$B^+ = 22,\ 23$	(3)
$B0 = 21$	(3)
$B^- = 20$	(3)
$C^+ = 18,\ 19$	(4)
$C0 = ..$	(3)