Solution 7

1. Use Lagrange multipliers to find the maximum and minimum values of the functions subject to the given constraint.

$$f(x,y) = 3x + y; \qquad x^2 + y^2 = 10$$

Solution. Let $g(x,y) = x^2 + y^2 - 10$. Then, we can apply the Lagrange multipliers method. We need to find $P = (x_0, y_0)$ satisfying

$$\nabla f(P) // \nabla g(P).$$

Note that $\nabla f(P) = (3,1)$ and $\nabla g(P) = 2(x_0, y_0)$. Hence, the ratios $x_0 : y_0 = 3 : 1$ are the same so that $x_0 = 3y_0$. Applying this relation to $g(x_0, y_0) = 0$ which is the constraint, we get $(x_0, y_0) = (3,1)$ or $(-3,-1)$. Among those two points $(3,1)$ gives $f(3,1) = 10$ which is larger than $f(-3,-1) = -10$. Hence, the maximum value is 10 at the point $(3,1)$ and the minimum value is -10 at the point $(-3,-1)$.

Answer. Max : 10, Min : -10.

2. Find the local maximum and minimum values and saddle point(s) of the function. Please give the value of maximum or minimum as well as the points at which the values are attained.

$$f(x,y) = xy(1 - x - y)$$

Solution. $f_x = y - 2xy - y^2 = y(1 - 2x - y)$, $f_y = x - 2xy - x^2 = x(1 - 2y - x)$. Making f_x and f_y both zero, there could be four possible cases : $(0,0)$, $(1,0)$, $(0,1)$, $(\frac{1}{3}, \frac{1}{3})$. Now, let's compute second partial derivatives. $f_{xx} = -2y$, $f_{xy} = 1 - 2x - 2y$, $f_{yy} = -2x$. Hence, $D = f_{xx}f_{yy} - f_{xy}^2 < 0$ for $(0,0)$, $(1,0)$, $(0,1)$. So those points are saddle points. However, for $(\frac{1}{3}, \frac{1}{3})$, it is positive and $f_{xx} < 0$. Thus a local maximum is attained at the point $(\frac{1}{3}, \frac{1}{3})$.

Answer. f has its local maximum $\frac{1}{27}$ at $(\frac{1}{3}, \frac{1}{3})$ and the saddle points are $(0,0)$, $(1,0)$, $(0,1)$.

3. (10pts) Find the absolute maximum and minimum values of

$$f(x,y) = x^4 + y^4 - 4xy + 2$$

on the set $D = \{(x,y) : 0 \le x \le 3, 0 \le y \le 2\}$.

Solution. First of all, we shall find all the critical points lying in the interior of D which can be represented as a set $\{(x,y) : 0 < x < 3, 0 < y < 2\}$. The first partial derivatives are

$$f_x = 4(x^3 - y), \ f_y = 4(y^3 - x).$$

A critical point (x,y) should satisfy $y = x^3$ and $x = y^3$ so that $y = y^9$. Hence, $y = 0$ or $y^8 - 1 = 0$. The latter equation has only two solutions $y = \pm 1$. In each cases, x is obtained as the same value as y(, surprisingly). There are three critical points $(-1,-1)$, $(0,0)$, $(1,1)$. Among them, there is only one point lying in the interior of D ; $(1,1)$. Note that $f(1,1) = 0$.

Now, let's check the boundary. The boundary can be splitted into 4 pieces ;

$$(0, y) : 0 \le y \le 2, \qquad (x, 0) : 0 \le x \le 3, \qquad (3, y) : 0 \le y \le 2, \qquad (x, 2) : 0 \le x \le 3$$

For each pieces, we have a function and some range

$$y^4 + 2 : 0 \le y \le 2, \qquad x^4 + 2 : 0 \le x \le 3, \qquad y^4 - 12y + 83 : 0 \le y \le 2, \qquad x^4 - 8x + 18 : 0 \le x \le 3$$

For the first two functions, maximum (or minimum) is attained when x or y is maximum (or minimum). Hence, we have two possible maximum values : $2^4 + 2$ and $3^4 + 2$, and two minimum values : 2 and 2.

$y^4 - 12y + 83$ has its critical point at $y = 3^{1/3} < 2$. The value at the point is $3^{1/3}(3 - 12) + 83 = 83 - 9 \cdot 3^{1/3}$. Values at the endpoints of the range of y are 83 ($y = 0$) and 75 ($y = 2$). Similarly, $x^4 - 8x + 18$ has its critical point at $x = 2^{1/3} < 3$. The value at that point is $2^{1/3}(2 - 8) + 18 = 18 - 6 \cdot 2^{1/3}$. Values at the endpoints of the range of x are 18 ($x = 0$) and 75 ($x = 3$).

Consequently, all the possibly maximum or minimum values we have found are located between $f(1, 1) = 0$ and $f(3, 0) = 83$.

Answer. Max : 83, Min : 0.

Letter grade for Quiz 7

$$\begin{aligned} A^+ &= 30 && (1) \\ A0 &= 28 && (4) \\ A^- &= 25 && (2) \\ B^+ &= 24 && (4) \\ B0 &= 22,\ 23 && (5) \\ B^- &= 20,\ 21 && (3) \\ C^+ &= 17,\ 18 && (4) \\ C0 &= .. && (5) \end{aligned}$$