Solution 1

1. Find dy/dx and d^2y/dx^2. For which values of t is the curve concave upward?

(a) $x = \cos 2t$, $y = \sin 2t$

Solution. First,

$$\frac{dx}{dt} = -2\sin 2t, \quad \frac{dy}{dt} = 2\cos 2t.$$

Using the formula

$$dy/dx = \frac{dy/dt}{dx/dt}, \tag{1}$$

we get $dy/dx = -\frac{\cos 2t}{\sin 2t} = -\cot 2t$.

The meaning of d^2y/dx^2 is

$$\frac{d^2y}{dx^2} = \frac{d(dy/dx)}{dx}.$$

Since we know what dy/dx is in terms of t, we can apply the above formula (1) again.

$$\frac{d(dy/dx)}{dx} = \frac{d(dy/dx)/dt}{dx/dt}.$$

Finally, $d^2y/dx^2 = \frac{2\csc^2 2t}{-2\sin 2t} = -\csc^3 2t$. When $\csc 2t < 0$, $d^2y/dx^2 > 0$ holds (or equivalently, the curve is concave upward). It is equivalent to $\sin 2t < 0$, or $2t = 2n\pi + \theta'$ where $\pi < \theta' < 2\pi$.

Answer. $dy/dx = -\cot 2t \ (= -\frac{\cos 2t}{\sin 2t})$
$d^2y/dx^2 = -\csc^3 2t (= -\frac{1}{\sin^3 2t})$
The curve is concave upward for $t = n\pi + \theta$
where n is an integer and $\frac{\pi}{2} < \theta < \pi$.

(b) $x = e^{t^2}$, $y = e^{3t+2}$

Solution. Similar to **1.(a)**, calculate dx/dt and dy/dt first and then use the formula (1).

$$\frac{dx}{dt} = 2te^{t^2}, \quad \frac{dy}{dt} = 3e^{3t+2}, \quad \frac{dy}{dx} = \frac{3}{2t}e^{-t^2+3t+2}.$$

Next, we get

$$\frac{d(dy/dx)}{dt} = -\frac{3}{2t^2}e^{-t^2+3t+2} + \frac{3}{2t}\cdot(-2t+3)e^{-t^2+3t+2} = -\frac{3(2t^2-3t+1)}{2t^2}e^{-t^2+3t+2}$$

$$\frac{d^2y}{dx^2} = -\frac{3(2t^2-3t+1)}{4t^3}e^{-2t^2+3t+2}$$

Since $t^4 \geq 0$ and $e^{-2t^2+3t+2} > 0$, we can conclude that $d^2y/dx^2 > 0$ if and only if $-(2t^2-3t+1)t > 0$. It is equivalent to $t(2t-1)(t-1) < 0$.

Answer. $dy/dx = \frac{3}{2t}e^{-t^2+3t+2}$
$d^2y/dx^2 = -\frac{3(2t^2-3t+1)}{4t^3}e^{-2t^2+3t+2}$
The curve is concave upward when $t < 0$ or $\frac{1}{2} < t < 1$.

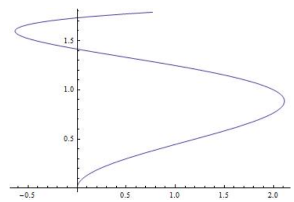
2. Find the area enclosed by the curve $x = t^3 - 5t^2 + 6t$, $y = \sqrt{t}$ and y-axis.

Solution. We need to calculate two integrations ; one for $0 \le t \le 2$, the other for $2 \le t \le 3$. And then, take their absolute values (to make them positive) and add those two positive numbers. The result would be the area enclosed by the curve defined and y-axis.

Here, the value 2 and 3 comes from the equation $x = t^3 - 5t^2 + 6t = 0$. There are three roots for that equation, $t = 0$, $t = 2$, $t = 3$. $t = 0$ represents for $(0, 0)$, $t = 2$ represents for the middle point on y-axis in the figure below, and $t = 3$ represents for the highest point on y-axis.

$$\int_0^2 y(t)x'(t)dt = \int_0^2 \sqrt{t}(3t^2 - 10t + 6)dt = \int_0^2 (3t^{5/2} - 10t^{3/2} + 6t^{1/2})dt = \frac{6}{7}2^{7/2} - 4 \cdot 2^{5/2} + 4 \cdot 2^{3/2} - 0 = -\frac{8}{7}\sqrt{2}.$$

Hence, we know that the area of the below is $\frac{8}{7}\sqrt{2}$.



$$\int_2^3 y(t)x'(t)dt = \left(\frac{6}{7}t^{7/2} - 4t^{5/2} + 4t^{3/2}\right)\Big|_2^3 = \frac{6}{7}3^{7/2} - 4 \cdot 3^{5/2} + 4 \cdot 3^{3/2} - (-\frac{8}{7}\sqrt{2}) = -\frac{6}{7}\sqrt{3} + \frac{8}{7}\sqrt{2} > 0$$

Hence, the area of the upper side area is $\frac{8}{7}\sqrt{2} - \frac{6}{7}\sqrt{3}$. Therefore, the total area is $\frac{16}{7}\sqrt{2} - \frac{6}{7}\sqrt{3}$.

Answer. $\frac{16}{7}\sqrt{2} - \frac{6}{7}\sqrt{3}$

3. Find the exact length of the curve,

$$x = t \sin 2t, \quad y = t \cos 2t, \quad 0 \le t \le 1.$$

Solution. This is similar to the problem we've done in class. We can calculate

$$\frac{dx}{dt} = \sin 2t + 2t \cos 2t, \quad \frac{dy}{dt} = \cos 2t - 2t \sin 2t, \quad \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{4t^2 + 1}.$$

Now, we need to calculate

$$\int_0^1 \sqrt{4t^2 + 1}dt.$$

You may compute this using *trigonometric substitution*, but I shall compute using $2t = \frac{e^s - e^{-s}}{2}$ substitution. For $s = 0$, t becomes 0. For t to be 1, $e^s - e^{-s} = 4$, that is, $(e^s)^2 - 4(e^s) - 1 = 0$. So, $e^s = 2 + \sqrt{5} > 0$ and $s = \ln(2 + \sqrt{5})$. Now the integration we need to compute changes to

$$\int_0^{\ln(2+\sqrt{5})} \frac{e^s + e^{-s}}{2} \cdot \frac{e^s + e^{-s}}{4} ds. \quad (\text{Note that } dt = d(\tfrac{e^s - e^{-s}}{4})/ds = \frac{e^s + e^{-s}}{4}.)$$

Calculating that, we get

$$\frac{1}{8}\int_0^{\ln(2+\sqrt{5})} (e^{2s} + 2 + e^{-2s})ds = \frac{1}{8}\left(\frac{1}{2}e^{2s} + 2s - \frac{1}{2}e^{-2s}\right)\Big|_0^{\ln(2+\sqrt{5})}.$$

So, the exact length of the curve is $\frac{1}{8}\left(\frac{1}{2}(2 + \sqrt{5})^2 + 2\ln(2 + \sqrt{5}) - \frac{1}{2}(\sqrt{5} - 2)^2\right) = \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(2 + \sqrt{5})$.

Answer. $\frac{\sqrt{5}}{2} + \frac{1}{4}\ln(2 + \sqrt{5})$

Letter grade for Quiz 1

$$
\begin{aligned}
19.0 &< A^+ \\
17.5 &< A0 \leq 18.5 \\
16.0 &< A^- \leq 17.5 \\
15.0 &< B^+ \leq 16.0 \\
13.5 &< B0 \leq 15.0 \\
12.5 &< B^- \leq 13.5 \\
& C^+ \leq 12.5
\end{aligned}
$$