Solution 5

1. Find the limit, if it exists, or show that the limit does not exist.

a)

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$

Solution. If (x, y) approaches to $(0, 0)$ along the curve $x = y^3$, the value of the function is

$$\frac{y^3y^3}{y^6+y^6} = \frac{1}{2}.$$

However, if we send (x, y) to $(0, 0)$ along the curve $x = 2y^3$, the value of the function is

$$\frac{2y^3y^3}{4y^6+y^6} = \frac{2}{5}.$$

Hence, the value of $f(x, y) = \frac{xy^3}{x^2+y^6}$ depends on the way which (x, y) approaches to $(0, 0)$ by. Therefore, it has no limit.

Answer. No limit.

b)

$$\lim_{(x,y)\to(0,0)} \frac{xye^y}{\sqrt{x^2+y^2}}$$

Solution. Using the inequality

$$x^2 + y^2 \geq 2|xy|,$$

we get

$$0 \leq \left|\frac{xye^y}{\sqrt{x^2+y^2}}\right| \leq \frac{xye^y}{\sqrt{2|xy|}} = \frac{1}{\sqrt{2}}\sqrt{|xy|}e^y.$$

We know that $\frac{1}{\sqrt{2}}\sqrt{|xy|}e^y$ is continuous on the whole plane. Hence, taking $\lim_{(x,y)\to(0,0)}$ on the inequality above, we get

$$0 \leq \lim_{(x,y)\to(0,0)} \left|\frac{xye^y}{\sqrt{x^2+y^2}}\right| \leq \lim_{(x,y)\to(0,0)} \frac{1}{\sqrt{2}}\sqrt{|xy|}e^y$$
$$= \frac{1}{\sqrt{2}}\sqrt{|0\cdot 0|}e^0 = 0$$

This shows that

$$\lim_{(x,y)\to(0,0)} \frac{xye^y}{\sqrt{x^2+y^2}} = 0.$$

Answer. **0**

2. Find the first partial derivatives of the function.

a)

$$f(a, b) = a^{1/3}\ln b$$

Answer.

$$\frac{\partial f}{\partial a} = \frac{1}{3}a^{-2/3}\ln b$$
$$\frac{\partial f}{\partial b} = \frac{a^{1/3}}{b}$$

b)

$$\varphi(x, y, z, t) = \frac{\sin^2 x + \sin^2 z}{\cos y + \cos t}$$

Answer.

$$\frac{\partial \varphi}{\partial x} = \frac{2\sin x\cos x}{\cos y + \cos t}$$
$$\frac{\partial \varphi}{\partial y} = \sin y \cdot \frac{\sin^2 x + \sin^2 z}{(\cos y + \cos t)^2}$$
$$\frac{\partial \varphi}{\partial z} = \frac{2\sin z\cos z}{\cos y + \cos t}$$
$$\frac{\partial \varphi}{\partial t} = \sin t \cdot \frac{\sin^2 x + \sin^2 z}{(\cos y + \cos t)^2}$$

3. Find an equation of the tangent plane to the given surface at the specified point.

a)

$$z = 3\cos x - 2\sin y + 5, \quad (\pi, \tfrac{\pi}{2}, 0)$$

Solution. First of all, we set

$$f(x,y,z) = 3\cos x - 2\sin y - z,$$

then the equation becomes $f(x,y,z) = 0$. Note that the gradient vector of f at $(\pi, \frac{\pi}{2}, 0)$ is perpendicular to the tangent plane at $(\pi, \frac{\pi}{2}, 0)$. We have first partial derivatives

$$f_x = -3\sin x, \quad f_y = -2\cos y, \quad f_z = -1.$$

Hence, the normal vector is

$$\begin{aligned}&(f_x((\pi, \tfrac{\pi}{2}, 0)), f_y((\pi, \tfrac{\pi}{2}, 0)), f_z((\pi, \tfrac{\pi}{2}, 0))\\ =&(0,0,-1)\end{aligned}$$

Therefore, the equation for the tangent plane at $(\pi, \frac{\pi}{2}, 0)$ is

$$(0,0,-1)\cdot(x-\pi, y-\frac{\pi}{2}, z-0) = 0.$$

Answer. $z = 0$ (the xy-plane)

b)

$$z = -2\ln x + (y+1)^2 - 1, \quad (e, -3, 1)$$

Solution. In this case,

$$f(x,y,z) = -2\ln x + (y+1)^2 - 1 - z.$$

Hence,

$$f_x = -\tfrac{2}{x}, \quad f_y = 2(y+1), \quad f_z = -1.$$

Thus, the normal vector is

$$\begin{aligned}&(f_x((e,-3,1)), f_y((e,-3,1)), f_z((e,-3,1)))\\ =&(-\frac{2}{e}, -4, -1)\end{aligned}$$

The tangent plane has its equation as

$$(-\frac{2}{e}, -4, -1)\cdot(x-e, y+3, z-1) = 0.$$

Answer. $\frac{2}{e}x + 4y + z + 9 = 0$

Letter grade for Quiz 5

$$\begin{aligned}29 &< A^+\\ 26 &< A0 \le 29\\ 24 &< B^+ \le 26\\ 18 &< B0 \le 24\\ 15 &< B^- \le 18\\ &C^+ \le 15\end{aligned}$$