How to write down right proofs exactly

Find the limit, if it exists, or show that the limit does not exist.

a)

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

Proof that the limit does not exist

If (x, y) approaches to $(0, 0)$ along the curve $x = y^2$, the value of the function is constantly

$$\frac{y^2y^2}{y^4+y^4} = \frac{1}{2}.$$

However, if we send (x, y) to $(0, 0)$ along the curve $x = 2y^2$, the value of the function is constantly

$$\frac{2y^2y^2}{4y^4+y^4} = \frac{2}{5}.$$

Hence, the value of $f(x, y) = \frac{xy^2}{x^2+y^4}$ depends on the way by which (x, y) approaches to $(0, 0)$. Therefore, it has no limit.

b)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$$

Proof that the limit exists and is 0.

Using the inequality

$$x^2 + y^2 \geq 2|xy|,$$

we get

$$0 \leq \left|\frac{x^2y^2}{x^2+y^2}\right| \leq \frac{x^2y^2}{2|xy|} = \frac{1}{2}|xy|.$$

We know that $\frac{1}{2}|xy|$ is continuous on the whole plane. Hence, taking $\lim\limits_{(x,y)\to(0,0)}$ on the inequality above, we get

$$0 \leq \lim_{(x,y)\to(0,0)} \left|\frac{x^2y^2}{x^2+y^2}\right| \leq \lim_{(x,y)\to(0,0)} \frac{1}{2}|xy| = \frac{1}{2}|0\cdot 0| = 0$$

This shows that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0.$$