QUIZ 9 (30MINS, 40PTS)

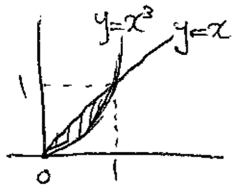
Please write down your name, SID, and solutions discernably.

Name : DongGyu Lim SID : Score :

1. **(10pts)** Evaluate the double integral,

$$\iint_D (x^2 + 2y)dA$$

where D is bounded by $y = x$, $y = x^3$, $x \geq 0$.



Let's use vertical segments, that is, fix x first $\Rightarrow$ $0 \le x \le 1$.

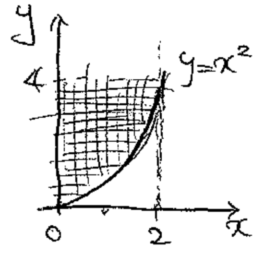
for each x, the range of y is $x^3 \le y \le x$.

$$\therefore \iint_D (x^2+2y)dA = \int_0^1 \int_{x^3}^{x} (x^2+2y)\,dy\,dx$$
$$= \int_0^1 (x^2y + y^2)\Big|_{x^3}^{x} dx$$
$$= \int_0^1 (x^3 + x^2 - x^5 - x^6)dx$$
$$= \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{7} = \frac{21+28-14-12}{84} = \frac{23}{84}$$

Answer

2. **(10pts)** Sketch the region of integration and change the order of integration.

$$\int_0^2 \int_{x^2}^4 f(x, y)dydx$$



The given region D can be written as

$\{(x,y) : 0 \le x \le 2,\ x^2 \le y \le 4\}$.

and this represents stacking vertical segments from the left to the right.

Now, changing the order of integration is to represent D as a process of stacking horizontal segments. $\Rightarrow$ $0 \le y \le 4$ is first fixed.

For each y, the range of x is $0 \le x \le \sqrt{y}$.

$\therefore$ Another representation for the integration is $\int_0^4 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy$

Answer

3. (10pts) Evaluate the given integral by changing to polar coordinates,

$$\iint_R \frac{x^2-y^2}{x^2+y^2}\,dA$$

where R is the region that lies between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ with $0<a<b$.

Since $dxdy = r\,dr\,d\theta$, $\iint_R \frac{x^2-y^2}{x^2+y^2}dA = \iint_R \frac{r^2(\cos^2\theta-\sin^2\theta)}{r^2(\cos^2\theta+\sin^2\theta)}\cdot r\,dr\,d\theta$.

Now, let's express R in polar coordinates. It is pretty simple. since $x^2+y^2=r^2$, $\therefore D=\{\overset{\text{polar coords}}{(r,\theta)} : a<r<b,\ 0\le\theta\le 2\pi\}$.

$\therefore \iint_R \frac{x^2-y^2}{x^2+y^2}dA = \int_a^b\int_0^{2\pi} r\cos 2\theta\, d\theta\, dr$ (Note that $\cos 2\theta = \cos^2\theta - \sin^2\theta$)

$= \int_a^b r\,dr \cdot \int_0^{2\pi}\cos 2\theta\, d\theta$.

$= \frac{1}{2}(b^2-a^2)\cdot \underbrace{\frac{1}{2}\sin 2\theta\Big|_0^{2\pi}}_{=0} = 0.$

Answer

4. (10pts) Evaluate $\iiint_E xe^{x^2+y^2+z^2}dV$, where E is the portion of the unit ball $x^2+y^2+z^2\le 1$ that lies in the first octant.

Let's use spherical coordinates,

First, $E=\{(x,y,z) : x^2+y^2+z^2\le 1, \underbrace{\text{first octant}}_{x,y,z\ge 0}\}$.

$=\{(r,\phi,\theta) : 0\le r\le 1$ and $r\sin\phi\cos\theta \ge 0 \Rightarrow 0\le\theta\le\pi/2$; $r\sin\phi\sin\theta\ge 0 \Rightarrow$; $r\cos\phi \ge 0 \Rightarrow 0\le\phi\le\pi/2\}$.

$\therefore \iiint_E xe^{x^2+y^2+z^2}dV = \int_0^1\int_0^{\pi/2}\int_0^{\pi/2} r\sin\phi\cos\theta\, e^{r^2}\cdot r^2\sin\phi\, d\theta\, d\phi\, dr$.

$= \int_0^1 r^3e^{r^2}dr \cdot \int_0^{\pi/2}\sin^2\phi\, d\phi \cdot \int_0^{\pi/2}\cos\theta\, d\theta$

$\int_0^{\pi/2}\sin^2\phi\, d\phi = \int_0^{\pi/2}\frac{1-\cos 2\phi}{2}d\phi = \pi/4 - 0$, $\quad \int_0^{\pi/2}\cos\theta\, d\theta = 1$

$= \pi/4 \cdot \int_0^1 r^3e^{r^2}dr$

let $s=r^2$ $\Rightarrow ds = 2r\,dr$ $\Rightarrow \pi/4\int_0^1 \frac{1}{2}se^s ds = \frac{\pi}{8}\int_0^1 se^s ds = \frac{\pi}{8}(s-1)e^s\Big|_0^1$

$= \frac{\pi}{8}\cdot 1 = \frac{\pi}{8}$ Answer