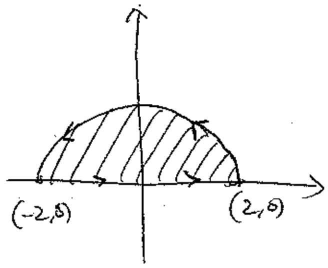
QUIZ 12 (20MINS, 30PTS)

Please write down your name, SID, and solutions discernably.

Name : DongGyu Lim SID : Score :

1. (10pts) A particle starts at the point $(-2, 0)$, moves along the x-axis to $(2, 0)$, and ten along the semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \langle x, x^3 + 3xy^2 \rangle$



Let $D = \{(x,y) : y \geq 0,\ x^2+y^2 \leq 4\}$

$= \{(r,\theta) : 0 \leq \theta \leq \pi,\ r \leq 2\}$. (Polar coordinates)

The work done on the particle is $\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$ given $\vec{F} = (P, Q)$

By Green's theorem, $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$= \iint_D (3x^2 + 3y^2 - 0)\, dA$

Change of variables

$= \int_0^2 \int_0^{\pi} 3r^2 \cdot r\,dr\,d\theta = \int_0^2 3r^3\,dr \int_0^{\pi} 1\,d\theta$

$= \frac{3}{4} \cdot 2^4 \cdot \pi = 12\pi$

Answer. 12π

2. (10pts) Find the curl and the divergence of the vector field.

$$\mathbf{F}(x, y, z) = \langle \ln x, \ln(xy), \ln(xyz) \rangle$$

P Q R.

$\text{curl}(\vec{F}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$= \langle \frac{1}{y} - 0, \cancel{0} - \frac{1}{x}, \frac{1}{x} - 0 \rangle$.

$\text{div}\,\vec{F} = P_x + Q_y + R_z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$.

Answer: $\text{curl}\,\vec{F} = \langle \frac{1}{y}, -\frac{1}{x}, \frac{1}{x} \rangle$

$\text{div}\,\vec{F} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

3. (10pts) Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

$$\mathbf{F}(x,y,z) = \underbrace{xyz^2}_{P}\mathbf{i} + \underbrace{x^2yz^2}_{Q}\mathbf{j} + \underbrace{x^2y^2z}_{R}\mathbf{k}$$

We need to check whether curl $\vec{F} = 0$

$$\text{curl}\,\vec{F} = \langle \underbrace{R_y}_{2x^2yz} - \underbrace{Q_z}_{2x^2yz},\ \underbrace{P_z}_{2xyz} - \underbrace{R_x}_{2xy^2z},\ \underbrace{Q_x}_{2xyz^2} - \underbrace{P_y}_{xz^2} \rangle = \langle 0,\ 2x^2yz - 2xyz,\ 2xyz^2 - xz^2 \rangle \neq 0$$

Hence, $\vec{F}$ is not conservative.

Answer. $\vec{F}$ is not a conservative vector field.