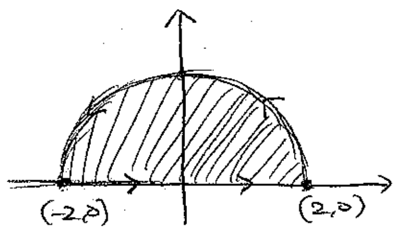
QUIZ 12 (20MINS, 30PTS)

Please write down your name, SID, and solutions discernably.

Name : DongGyu Lim SID : Score :

1. (10pts) A particle starts at the point $(-2, 0)$, moves along the x-axis to $(2, 0)$, and ten along the semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \langle x^2, x^2 + 2xy \rangle$



Let $D = \{(x,y) : y \geq 0,\ x^2+y^2 \leq 4\}$
$= \{(r,\theta) : 0 \leq \theta \leq \pi,\ r \leq 2\}$ in Polar coordinates.

The work done on the particle is $\int_C \vec{F} \cdot d\vec{r}$

By Greens Theorem, $\int_C \vec{F} \cdot dr = \int_C P\,dx + Q\,dy \overset{\text{Green's Thm}}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

(C is positively oriented)

$$= \iint_D (2x + 2y - 0)\,dA$$

Coordinate change

$$= \int_0^{2}\int_0^{\pi} 2r(\cos\theta + \sin\theta) \cdot r\,dr\,d\theta$$

$$= \int_0^2 2r^2\,dr \cdot \int_0^{\pi} (\cos\theta + \sin\theta)\,d\theta = \frac{16}{3} \cdot 2 = \frac{32}{3}.$$

$\sin\theta - \cos\theta \Big|_0^{\pi}$

Answer. $\frac{32}{3}$.

2. (10pts) Find the curl and the divergence of the vector field.

$$\mathbf{F}(x, y, z) = \langle \ln y, \ln(yz), \ln(xyz) \rangle$$

P Q R

$\operatorname{curl} \vec{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$

$= \langle \frac{1}{y} - \frac{1}{z},\ 0 - \frac{1}{x},\ 0 - \frac{1}{y} \rangle$

$\operatorname{div} \vec{F} = P_x + Q_y + R_z = 0 + \frac{1}{y} + \frac{1}{z}.$

Answer. $\operatorname{curl} \vec{F} = \langle \frac{1}{y} - \frac{1}{z}, -\frac{1}{x}, -\frac{1}{y} \rangle$

$\operatorname{div} \vec{F} = \frac{1}{y} + \frac{1}{z}$

3. (10pts) Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

$$\mathbf{F}(x,y,z) = ye^{-x}\mathbf{i} + e^{-x}\mathbf{j} + 2z\mathbf{k}$$

P Q R

We need to check whether curl $\vec{F} = 0$ or not.

$$\text{curl}\,\vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle = \langle 0-0,\ 0-0,\ -e^{-x}-e^{-x} \rangle, \quad -e^{-x}-e^{-x} \neq 0$$

Since curl $\vec{F} \neq 0$, $\vec{F}$ is not conservative.

Answer. $\vec{F}$ is not a conservative vector field.