1. $f(x+iy)$ is holomorphic. $\Rightarrow$ Cauchy-Riemann equation.
$\overline{f}(x+iy)$ is holomorphic.

$f(x+iy) = u(x,y) + i v(x,y)$ is holo.

$\Rightarrow u_x = v_y$ & $u_y = -v_x$.

$\rightarrow u_x = (-v)_y$ & $u_y = -(-v)_x$.

$= -v_y \qquad = v_x$.

$\therefore \underbrace{u_x = 0, \; u_y = 0}, \; \underbrace{v_x = 0, \; v_y = 0}$.

$\Rightarrow u(x,y) =$ constant (real) $\qquad v(x,y)$: constant (real).

$\Rightarrow f(x+iy) =$ const $+ i$ const

$=$ const (complex)

$f = c$ for some $c \in \mathbb{C}$.

2. (a) If $f = u + iv$ is holo, then v is a harmonic conjugate of u.

$\rightarrow$ Find v s.t. $v_x = -u_y$ & $v_y = u_x$.

$v_x = -4 + 6x^2 - 12y - 6y^2$

$v_y = \frac{1}{2} - 12x - 12xy$

$$v(x,y) = -4x + 2x^3 - 12xy - 6xy^2 + \underbrace{\text{function of } y}_{=: g(y)}.$$

$$v_y(x,y) = g'(y) - 12x - 12xy$$

$$= \frac{1}{2} - 12x - 12xy$$

$$\Rightarrow \quad g(y) = \frac{1}{2}y + C$$

$$\Rightarrow v(x,y) = -4x + \frac{1}{2}y - 12xy - 6xy^2 + 2x^3 + C.$$

$(C=0)$

(6) $f = u + iv$

$$f(x+iy) = \frac{x}{2} - 6x^2 + 4y - 6x^2y + 6y^2 + 2y^3 + i\left(-4x + \frac{1}{2}y - 12xy - 6xy^2 + 2x^3\right).$$

$$f(z) = (\quad)z + (\quad)z^2 + (\quad)z^3$$

$(\quad)z^3$

$$z^3 = (x+iy)^3 = x^3 + \cdots$$

$$\frac{1}{2}(x+iy) = \frac{1}{2}z$$

(c) From part b, $f(z) =$ polynomial in z

largest dom $= \mathbb{C}$.

3. D: closed disc, U: interior of D.

$$\oint_{\partial D} \frac{f(z)}{z-a} dz = 2\pi i \cdot f(a)$$

for any $\boxed{a \in D}$

when f is holomorphic on U.

(a) $$\oint_{|z-3|=2} \frac{f(z)}{z-3} \Rightarrow \text{Find } f(z) \text{ s.t.}$$

$$\frac{f}{z-3} = \frac{e^{-z^2}}{z^3 - pz^2 + 11z + 21}$$

$$\Rightarrow D = \{z : |z-3| \leq 2\}.$$

$$U = \{z : |z-3| < 2\}.$$

$$\| \quad (z-3)(\quad)$$

$$f(z) = \frac{e^{-z^2}}{\text{quad. poly.}}$$

Need to check is holo. on U.

$\Rightarrow 2\pi i \cdot f(3)$.

(6) $\oint_{|z-1|=2} \frac{\sin(z)}{z^2-4} dz$

"$a=1$" $\quad f(z) = \frac{\sin(z)}{z^2-4} \cdot \underline{(z-1)}$

$f(z)$ is not holomorphic

on $U = \{z \in \mathbb{C} : |z-1| < 2\}$.

$z=2$ is inside U.

So, $\frac{1}{z^2-4}$ is not holo. on U.

$\boxed{a=2}$ $\quad f(z) = \frac{\sin(z)}{z+2}$

$D = \{z \in \mathbb{C} : \cancel{|z-a| \le \square}\}$

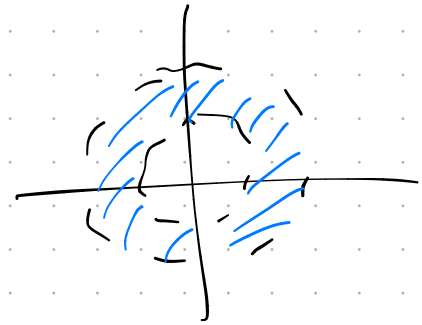
$|z-1| \le 2$.

4. The maximum modulus principle.

f : holo. on U (bdd)

$\Rightarrow$ $|f(z)|$ attains the max'm on ∂U.

$U = \{z \in \mathbb{C} : 2 < |z| < 3\}$.



Let $g(z) = \dfrac{f(z)}{z^2}$

is holo. on U.

$|f(z)| \le 16$ on $|z| = 2$.

$\Rightarrow \left|\dfrac{f(z)}{z^2}\right| \le 4$ on $|z| = 2$.

$|f(z)| \le 36$ on $|z| = 3$

$\Rightarrow \left|\dfrac{f(z)}{z^2}\right| \le 4$ on $|z| = 3$.

$\partial U = \{z : |z| = 2\} \cup \{z : |z| = 3\}$.

$\left|\dfrac{f(z)}{z^2}\right| \le$ max'm on the boundary

≤ 4.

5. (a) $z = z_0 + r \cdot e^{it}$

$dz = ? \cdots dt$

$\oint_{C_r} \dfrac{f(z)}{z - z_0}\, dz = \int_0^{2\pi} \cdots\, dt$.

(b) "$\underline{r \to 0}$"

$$\int_0^{2\pi} \sim\sim\sim \, dt \xrightarrow[r \to 0]{} \int_0^{2\pi} \underline{f(z_0)} \cdot i\,dt$$

$$= 2\pi i \cdot f(z_0).$$

(c) "$\underline{\text{commutativity of operations}}$"