S $\quad \|\cdot\|_2 = \sqrt{\langle \cdot\,, \cdot \rangle}$

$$\langle f, g \rangle := \int_{-\pi}^{\pi} fg \, dx$$

$$T_n(x) = c_0 + \sum_{m=1}^{n} a_m \cos mx + \sum_{m=1}^{n} b_m \sin mx$$

$$\|T_n\|_2 = \sqrt{\langle T_n, T_n \rangle} = \sqrt{\int_{-\pi}^{\pi} T_n \cdot T_n \, dx}$$

$$\| \quad \left(c_0 + \sum_{m=1}^{n} a_m \cos mx + \sum_{m=1}^{n} b_m \sin mx\right)\left(\;''\;\right)$$

$$\int_{-\pi}^{\pi} T_n^2 \, dx = \int_{-\pi}^{\pi} c_0^2 \, dx + \sum_{m=1}^{n} \int_{-\pi}^{\pi} a_m^2 \cos^2 mx \, dx + \sum_{m=1}^{n} \int_{-\pi}^{\pi} b_m^2 \sin^2 mx \, dx$$

$$+ \int \big(2 \cdot c_0 a_m \cos mx + 2 c_0 b_m \sin mx + 2 a_m \cos mx \, b_{m'} \sin m'x + 2 a_m \cos mx \, a_{m'} \cos m'x \big) = 0$$

$$\int_{-\pi}^{\pi} \cos lx \sin kx \, dx = 0$$

$$\int_{-\pi}^{\pi} \cos lx \cos kx \, dx = 0 \quad \text{if } l \neq k$$

as a result

$$\cos lx \cos kx = \frac{1}{2}\left[\cos(l+k)x + \cos(l-k)x\right]$$

$\uparrow$ use this

Consequently

$$= c_0^2 \int_{-\pi}^{\pi} 1\,dx + \sum_{m=1}^{n} a_m^2 \int_{-\pi}^{\pi} \cos^2 mx\,dx + \sum_{m=1}^{n} b_m^2 \int_{-\pi}^{\pi} \sin^2 mx\,dx$$

$$\sin^2 mx = 1 - \cos^2 mx$$

$\sqrt{\text{your computation}}$

$$= \text{const.} \sqrt{\underbrace{c_0^2 + \sum (a_m^2 + b_m^2)}_{\text{const}}}$$

6 1) Weierstraß M-test

$$a_0(x) + a_1(x) + \cdots + a_n(x) + \cdots$$

$$\sum_{i=0}^{n} \max |a_i(x)| < \infty$$

$\Rightarrow$ sum converges absolutely

Fourier series of f

$= \frac{a_0}{2} + \sum_{m=1}^{n} a_m \cos mx + \sum_{m=1}^{n} b_m \sin mx$

$|a_n \cos nx + b_n \sin nx|$

$\leq |a_n \cos nx| + |b_n \sin nx|$

M-test $\leq |a_n| + |b_n|$

$\Rightarrow$ Fourier series converges absolutely and uniformly

2) " to f " (Theorem 7) pointwise

Fourier series converges to the original ftn if f is cont.

7. $\varepsilon = \frac{1}{n}$, $\exists T_n$ st $|f(x) - T_n(x)| < \frac{1}{n}$ $\forall x \in \mathbb{R}$

$\Rightarrow T_n(x)$ converges to $f(x)$ pointwise for any $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon}$.

Then $\forall n \geq N$ $\quad |f(x) - T_n(x)| < \frac{1}{n} < \varepsilon$ $\quad \left(\frac{1}{N} < \varepsilon\right)$

$\forall x \in \mathbb{R}$.

$\therefore$ f is the uniform limit of trig polys.

(b) $\{T_n\}$ converges uniformly to f

continuous

$\xRightarrow{\text{thm}}$ f is conti.

$$\begin{array}{ccc} f(x) & \overset{?}{=} & f(x+2\pi) \\ \| & & \| \\ \lim\limits_{n\to\infty} T_n(x) & \underset{b/c}{=} & \lim\limits_{n\to\infty} T_n(x+2\pi) \end{array} \quad \forall x \in \mathbb{R}.$$

T_n is 2π-periodic

8. (Ch 6.2 Example 1)

~~$u(x,t) = v(x) \cdot w(t)$~~

~~$\Rightarrow v(x) = \sin nx, \quad w(t) = e^{-4n^2t}$~~

~~Guess: $u(x,t) = \sum_{n=1}^{\infty} c_n \sin nx \, e^{-4n^2t}$~~

& 1st & 2nd conditions.

3rd condition: $t=0$

$\Rightarrow c_1 = 1 \quad c_5 = -3$

$\therefore\ u(x,t) = \sin x \cdot e^{-4t} - 3\sin 5x \cdot e^{-100t}$.

q. $u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi c t}{L} + b_n \sin \frac{n\pi c t}{L} \right)$

$\sin \frac{n\pi x}{L}$

$L = 1 \quad c = \sqrt{\rho} = 3$.

$a_n = \frac{2}{L} \int_0^L u(x,0) \sin \frac{n\pi x}{L}\, dx$

$b_n = \frac{2}{n\pi c} \int_0^L u_t(x,0) \sin \frac{n\pi x}{L}\, dx$

$$\int_0^1 \sin m\pi x \sin n\pi x \, dx = 0$$

if $m \neq n$.

You can compute a_n's and b_n's.

$\Rightarrow$ there are 3 terms in the final answer.