1. $E \subseteq \mathbb{R}$ $\quad f_n : E \to \mathbb{R}$. $\quad f_n \to f$ uniform convergence

$\iff$ $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

$\sup A = \max A$.

$A = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$

$\sup A = 1$, ~~max A~~

($\Leftarrow$) Suppose $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

$$\|f_n - f\|_\infty = \sup_{x \in E} |f_n(x) - f(x)|$$

This implies $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ st $\forall n \geq N$

$$|f_n(x) - f(x)| \leq \sup_{x \in E} |f_n(x) - f(x)| < \varepsilon.$$

$a_n \to \#$ as $n \to \infty$

defn: $\forall \varepsilon > 0$ $\exists N > 0$ st $\forall n \geq N$ $|a_n - \#| < \varepsilon$

$\forall x \in E$.

Consider this new part

Uniformly converge

($\Rightarrow$) U.C $\Rightarrow$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ st $\forall n \geq N$, and $\boxed{\forall x \in E}$

$$\boxed{|f_n(x) - f(x)| < \varepsilon/2}$$

$$\Downarrow$$

$$\|f_n - f\|_\infty = \sup_{x \in E} |f_n(x) - f(x)| \leq \varepsilon/2$$

Rephrase: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t $\forall n \geq N$

$$\sup_{x \in E} |f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon.$$

$$\Rightarrow \|f_n - f\|_\infty \to 0 \text{ as } n \to \infty.$$

2. $f_n(x) = x^2 \cdot e^{-nx}$.

(a) Converges pointwise

fix x and then send n to ∞.

$x = 0 \Rightarrow f_n(0) = 0$ it converges to 0.

$x \neq 0 \implies f_n(x) = \dfrac{x^2}{(e^x)^n} = \dfrac{\#}{(\#)^n} \to 0$.
$(x>0)$ $\qquad$ $\# > 1$

(f_n) converges to the zero ftn pointwise.

(b) Converges uniformly. (Let's use Problem 1).

$\iff \|f_n - f\|_\infty \to 0$ as $n \to \infty$.

(1) f should be the zero ftn.
(from (a), this is obvious)

$\implies \|f_n - f\|_\infty = \|f_n\|_\infty = \sup_{x\in[0,\infty)} \left| \dfrac{x^2}{e^{nx}} \right| \xrightarrow{?} 0$ as $n\to\infty$.

(2) Compute $\sup_{x\in[0,\infty)} \left| \dfrac{x^2}{e^{nx}} \right|$ for a fixed $n \in \mathbb{N}$.

Want to find the maxim of $\dfrac{x^2}{e^{nx}}$ on $x \in [0,\infty)$.

Intuition: 1. $x=0 \implies f_n(x) = 0$.

2. it is a positive-valued ftn.

3. as $x \to \infty$ numerator: poly
denominator: exp

$\implies \dfrac{x^2}{e^{nx}}$ will go to 0.

$$\left(\frac{x^2}{e^{nx}}\right)' = \frac{2x\cdot e^{nx} - x^2\cdot n e^{nx}}{e^{2nx}} = \frac{1}{e^{nx}}\left(2x - x^2 n\right)$$

$$= \frac{x \cdot n}{e^{nx}}\left(\frac{2}{n} - x\right).$$

$$\left.\begin{aligned} x < \tfrac{2}{n} &\Rightarrow \text{deriv.} > 0 \\ x = \tfrac{2}{n} &\Rightarrow \text{deriv.} = 0 \\ x > \tfrac{2}{n} &\Rightarrow \text{deriv} < 0. \end{aligned}\right\} \Rightarrow x = \frac{2}{n} \text{ is the unique maximum.}$$

the unique zero of deriv.

$= \text{supremum}$

$$\therefore \sup_{x\in[0,\infty)} \left|\frac{x^2}{e^{nx}}\right| = \text{when } x = \frac{2}{n}$$

$$= \boxed{\frac{4}{e^2} \cdot \frac{1}{n^2}}.$$

goes to 0 as $n \to \infty$

3. (a) LHS: $\lim_{n\to\infty}\left(\lim_{x\to 0} f_n(x) = f_n(0) = 1\right) = 1.$ (conti.)

$\#$

RHS: $\lim_{x\to 0}\left(\lim_{n\to\infty} f_n(x) = \begin{cases} \cancel{x = 0 \Rightarrow 1} \\ x \neq 0 \Rightarrow 0 \end{cases}\right) = 0.$

(b) Thm. On $[a, b]$, if $\{f_n\} \to f$ uniformly and f_n's are all continuous. Then f is continuous.

$f_n(x)$: continuous. Suppose that $\{f_n\}$ converges uniformly to a function, say f. On $[-\frac{1}{2}, \frac{1}{2}]$.

Then, by thm, $f(x)$ is conti. at 0.

So, $\lim_{x \to 0} f(x) = f(0)$

$$\lim_{x\to 0} f(x) = \lim_{n\to\infty} f_n(x) \quad \text{RHS}$$

$$f(0) = \lim_{n\to\infty} f_n(0) = \underbrace{\lim_{n\to\infty} \lim_{x\to 0} f_n(x)}_{\text{LHS}}.$$

RHS $\neq$ LHS

Contradiction.

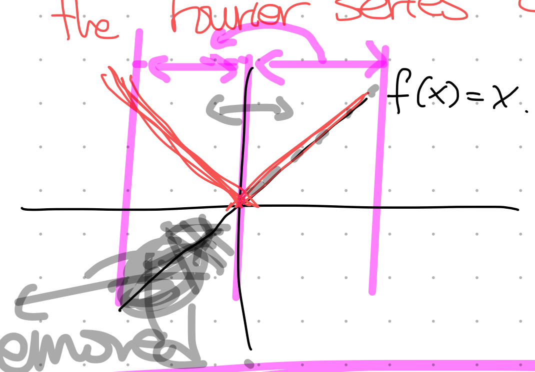
4. Fourier series and Fourier cosine series

$\|$

(Fourier sine series in this problem b/c $f(x)=x$ is an odd ftn.

Even if $f(x)=x$ on $x\in[-\pi,\pi]$,

Fourier cosine series is the Fourier series of



Fourier series of $f(x)=x$: odd ftn.

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx = 0$ $\quad \int_{-\pi}^{\pi} \text{odd}\cdot\text{even} = \text{odd} = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx \qquad \int u v' = uv - \int u' v$$

$u = x$, $v' = \sin nx$

$$= \frac{1}{\pi}\left(x\cdot\frac{\cos nx}{-n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\cos nx}{-n}\,dx \right)$$

$\cos n\pi = (-1)^n$

$$= \frac{1}{\pi}\left(\pi\cdot\frac{(-1)^n}{-n} - (-\pi)\cdot\frac{(-1)^n}{-n} - \frac{\sin nx}{-n^2}\Big|_{-\pi}^{\pi} \right)$$

$$= \frac{1}{\pi}\left(2\pi\cdot\frac{(-1)^{n-1}}{n} - 0 \right) = \frac{2}{n}(-1)^{n-1}.$$

$\Rightarrow$ X's Fourier series is

$$\sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n-1}\cdot \sin nx.$$

by some thm $x =$ Fourier series of x on $(-\pi, \pi)$.

$x = \pi/2$

$$\Rightarrow \frac{\pi}{2} = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n-1}\sin\frac{n\pi}{2}.$$

$$= 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots -$$

Useless for this problem.

$\Rightarrow$ Fourier cosine series.