5. $u(x,0) = \begin{cases} 0 & x < 0 \\ 10 - x & x \geq 0 \end{cases}$

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t)\, u(y,0)\, dy$$

This is not applicable.

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} u(y,0)\, dy$$

$$= \int_{0}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} (10-y)\, dy$$

$\boxed{(x-y)^2} \rightsquigarrow y^2$

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$$

$z = x - y$

Substitute $x-y$ by z ("=" $y = x - z$)

$$= \int_{x}^{-\infty} \frac{1}{\sqrt{4k\pi t}} \cdot e^{-z^2/4kt} \cdot (10 - x + z)\, d(-z)$$

$$= \int_{-\infty}^{x} \text{''} \cdot e^{-z^2/4kt} \cdot (10 - x + \boxed{z})\, dz$$

$$\int e^{-z^2/4kt} \cdot z\, dz = \boxed{\text{const.}} \cdot e^{-\frac{z^2}{4kt}}$$

$$\underbrace{(10-x)}_{\substack{\varphi \\ \text{"const" w.r.t. } z}} \cdot \underbrace{\int_{-\infty}^{x} e^{-z^2/4kt}} = \underbrace{\int_{-\infty}^{0}}_{\text{explicit number}} + \underbrace{\int_{0}^{x}}_{\underline{\text{Erf}}}$$

$$\Rightarrow u(x,t) = \text{const} \cdot e^{-z^2/4kt} + \boxed{\text{some number}} \cdot (10-x)$$

$$+ \text{Erf}(\quad) \cdot (10-x).$$

6. $u(x,0) = x^2$.

(a) if u satisfies $u_t = k u_{xx}$

using Clairaut's thm, $\underset{\overset{\|}{v_t}}{u_{xxxt}} = k \cdot \underset{\overset{\|}{v_{xx}}}{u_{xxxxx}}$.

$$v(x,0) = u_{xxx}(x,0) = (x^2)''' = 0.$$

(b) Fact: if $v(x,0) = 0$, then $v(x,t) \equiv 0$.

$\Rightarrow u_{xxx}(x,t) \equiv 0$.

$\Rightarrow u(x,t) = C_2 \cdot x^2 + C_1 \cdot x + C_0$.

C_2, C_1, C_0 are functions of t.

① $u_t = k u_{xx}$

x^2 $\qquad$ x

$$\Rightarrow \text{(LHS)}\ \overset{0}{\overset{\|}{C_2'(t)}}\cdot x^2 + \overset{0}{\overset{\|}{C_1'(t)}}\, x + \boxed{C_0'(t)}$$

$$\|$$

$$\text{(RHS)}\ \boxed{k \cdot C_2(t) \cdot 2}$$

constant part

② $u(x,0) = x^2$

$$\Rightarrow \underset{1}{\underset{\|}{C_2(0)}}\, x^2 + \underset{0}{\underset{\|}{C_1(0)}}\cdot x + \underset{0}{\underset{\|}{C_0(0)}} = x^2$$

$\Rightarrow$ Graph info about $C_2(t), C_1(t), C_0(t)$

$\Rightarrow$ can solve for them.

7. Conservation of energy for the wave eqn.
(Dirichlet condition or Neumann condition)

Statement: Let u be a solution for the wave eqn.

$$u_{tt} = c^2 \Delta u \quad \text{w/ } u(x,0) = \phi(x)$$
$$\text{and } u_t(x,0) = \psi(x)$$
$$\text{for } x \in U,\ t > 0.$$

Moreover, assume that u satisfies the homogeneous D or N boundary condition.

Let's define the energy function $E(t)$ as follows:

$$E(t) := \int_U \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}\left(\sum_{i=1}^{n} u_i^2\right)\right) dV$$

where u_i denotes $\frac{\partial u}{\partial x_i}$. Then, $E(t)$ is constant.

pf. $\frac{d}{dt}E(t) = \int_U \frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2 + \frac{c^2}{2}\left(\sum u_i^2\right)\right) dV$

$$= \int_U \Big(\underbrace{u_t u_{tt}}_{\| } + c^2 \sum_{i=1}^{n} u_i \cdot u_{it}\Big) dV$$

$c^2 \Delta u = c^2 \sum_{i=1}^{n} u_{ii}$ (u_{ii} means $\frac{\partial^2 u}{\partial x_i^2}$)

$$= \int_U c^2 \sum_{i=1}^{n} \left(u_t \cdot u_{ii} + u_i u_{it}\right) dV$$

↙ Clairaut

$$= \int c^2 \cdot \sum_{i=1}^{n} \left(u_t \cdot u_i\right)_i dV$$

$$= c^2 \cdot \int_U \sum_{i=1}^{n} \frac{\partial(u_t u_i)}{\partial x_i} dV_n \leftarrow \text{n-dim'l volume factor}$$

divergence thm

$$= c^2 \cdot \int_{\partial U} \boxed{(u_t u_1, \dots, u_t u_n) \cdot \vec{n}} \, dV_{n-1}$$

|

1) (hom.) Dirichlet boundary

$\Rightarrow u_t(x,t) = 0$ for $x \in \partial U$.

$$= c^2 \cdot \int_{\partial U} (0, 0, \dots, 0) \cdot \vec{n} \, dV_{n-1}$$

$$= 0.$$

2) (hom) Neumann boundary.

$$\underline{\operatorname{grad} u_t \cdot \vec{n} = 0}$$

$$\frac{\partial u_t}{\partial \nu} = 0$$

ν : normal direction to ∂U.

$$(u_1, \dots, u_n) \cdot \nu = 0.$$

$$= c^2 \cdot \int_{\partial U} 0 \, dV_{n-1} = 0.$$

$\frac{d}{dt} E(t) - \dots = \dots 0 \Rightarrow E(t)$ is constant.

8. Suppose u_1 and u_2 are two sol'ns.

Define $v(x,t) := u_1(x,t) - u_2(x,t)$.

Then, $v_t = k v_{xx}$

$$\boxed{v(x,0) = u_1(x,0) - u_2(x,0) = 0}$$

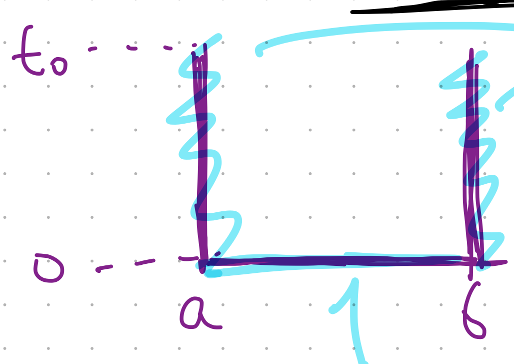
$v_t(x,t) =$

This is a heat eqn for a bounded interval $[a,b]$.

$\Rightarrow$ Use maximum principle!

$\underline{v(x,t) \le 0}$ for all $x \in [a,b]$, $t > 0$.

Dirichlet boundary condition.



but we can also consider

$u_2 - u_1 = -v$

$\Rightarrow -v(x,t) \le 0$.

$\Longrightarrow v(x,t) = 0$

$\dots \Rightarrow u_1 = u_2$

(OR $\underline{u_1 - u_2 = 0}$)

$\Longrightarrow$ the sol'n is unique

9. Define $w(x,t) = v(x,t) - u(x,t)$.

w is a sol'n of the heat eqn.

↳ ① $w_t = k w_{xx}$.

② $w(a,t) = w(b,t) = 0$.

This is true only for $[a,b]$.

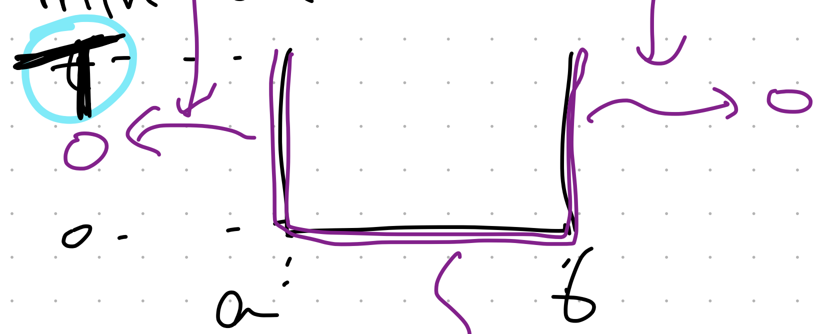
$(-\infty, \infty)$

$\Rightarrow$ this will fail.

Tychonoff's solution.

minimum principle.

$\Rightarrow$ min $w(x,t)$ is obtained on



0 $(w(x,t) = v(x,t) - u(x,t))$

$\Rightarrow$ $\underline{w(x,t) \geq 0}$. for any $t < T$.

but T is arbitrary $\Rightarrow$ $w(x,t) \geq 0$ $\forall t > 0$.

"homogeneous heat equation"

$\Rightarrow$ u satisfies ① $u_t = k u_{xx}$

② $u(x,0) = \phi(x)$

③ $u(a,t) = \psi_a(t)$ $u(b,t) = \psi_b(t)$.