1. Divergence theorem = Green's theorem

① $$\iint_{\Omega} (P_y - Q_x)\,dS = \int_{\partial\Omega} P\,dx + Q\,dy$$

→ over here

$u(x,t)$

②' c: the specific heat ρ: density u: heat $\Rightarrow E = c\cdot\rho\cdot u$.

② Fourier's law : $-k_0\,(\text{gradient } u)\cdot\vec{n}$ ∇u

will be the amount of the heat escaping from inside.

③ Philosophy : Energy conservation.

$$\frac{d}{dt}\iint_{\Omega} c\rho u\,dS \quad \text{vs} \quad \int_{\partial\Omega} -k_0 \nabla u\cdot\vec{n}\,ds$$

negative of $=$

measures the amount of energy escaping

③ the same

x's function

$$-\frac{d}{dt}\iint_{\Omega} c\rho u \, dS = -k_0 \int_{\partial\Omega} \nabla u \cdot \vec{n} \, ds$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

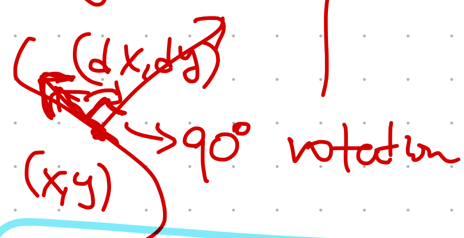
$$-\iint_{\Omega} c\rho \frac{\partial}{\partial t} u \, dS \qquad -k_0 \int_{\partial\Omega} (u_x, u_y)\cdot(\quad)\, ds$$

$$(dy, -dx)$$

$$\parallel$$

$$-\iint_{\Omega} c\rho u_t \, dS$$

why?



|| Apply divergence theorem

$$\Rightarrow \iint_{\Omega} \left(u_t - k(u_{xx}+u_{yy})\right) dS = 0.$$

Ω can be any bounded region.

$$\Rightarrow u_t = k \cdot (u_{xx}+u_{yy}).$$

2. $f*g\,(x) = \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy$

$h*h(x) = \int_{-\infty}^{\infty} (x-y)\,e^{-(x-y)^2}\cdot y\cdot e^{-y^2}\,dy.$

fixed

$-\infty < y < \infty$

$x-y$ & y are sym. w.r.t. $\frac{x}{2}$.

$y = \frac{x}{2} + z \Rightarrow -\infty < z < \infty$

$dy = dz$

$$\boxed{\frac{x}{2} - z \ \& \ \frac{x}{2} + z}$$

$$= \int_{-\infty}^{\infty} \left(\frac{x}{2} - z\right) e^{-\left(\frac{x}{2}-z\right)^2} \cdot \left(\frac{x}{2}+z\right)\cdot e^{-\left(\frac{x}{2}+z\right)^2}\,dz$$

$$= \int_{-\infty}^{\infty} \left(\frac{x^2}{4} - z^2\right) e^{-\frac{x^2}{2} - 2z^2}\,dz$$

$$= \int \frac{x^2}{4} e^{-\frac{x^2}{2}} \cdot e^{-2z^2} + \int -z^2 \cdot e^{-\frac{x^2}{2}} \cdot e^{-2z^2}\,dz.$$

$$= \frac{x^2}{4}\cdot e^{-\frac{x^2}{2}} \underbrace{\int_{-\infty}^{\infty} e^{-2z^2}\,dz}_{\sqrt{\frac{\pi}{2}}} - e^{-\frac{x^2}{2}} \int z^2 \cdot e^{-2z^2}\,dz$$

1st use $\int uv' = uv - \int u'v$

$u = z$

$v = z\cdot e^{-2z^2}$ $\quad v$

$\int = -\frac{1}{4} e^{-2z^2}$

2nd Define

$\frac{d}{dt}(f(t)) = \int_{-\infty}^{\infty} e^{-tz^2}\,dz$

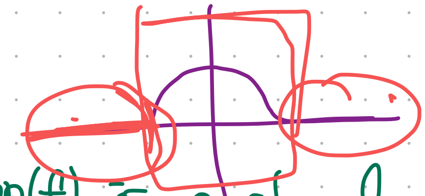
$f'(t) = \int_{-\infty}^{\infty} -z^2 \cdot e^{-tz^2}\,dz = \frac{d}{dt}\sqrt{\frac{\pi}{t}}$

3. $\text{supp}(f) \neq \{x\in\mathbb{R} : f(x)\neq 0\}$

$\int f$

$= \overline{\{x\in\mathbb{R} : f(x)\neq 0\}}$

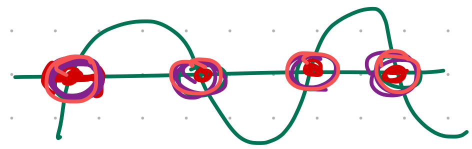
supp(f) is a closed subset in $\mathbb{R}$



$\Downarrow$

$\text{supp}(f) + \text{supp}(g)$ is also closed.

$f(x)$'s graph =



then $\text{supp}(f) = \mathbb{R}$.

not interested in zeros of f.

but interested in $\longleftrightarrow$ where f vanishes.

$x_0 \notin \text{supp}(f) \iff \exists\, \varepsilon > 0$ s.t. f vanishes on $B(x_0,\varepsilon)$. This criterion

$$f*g\,(x) = \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy.$$

$\notin (\ \ )$ $=$ $\in (\ \ )^c$

$\text{supp}(f*g) \subseteq \text{supp}(f) + \text{supp}(g)$

$\iff \underbrace{(\qquad)^c}_{\text{open}} \supseteq \underbrace{(\qquad)^c}_{\text{open}}$

Enough to show that if $\underbrace{x_0 \in (\text{supp} f + \text{supp} g)^c}_{\text{open set}}$

then $x_0 \in (\text{supp}(f*g))^c$.

$\rightarrow \exists\, \varepsilon > 0$ s.t. $B(x_0,\varepsilon) \subseteq (\text{supp} f + \text{supp} g)^c$.

we want to prove that inside $B(x_0,\varepsilon)$, $f*g$ vanishes.

$\Rightarrow x_0 \in \text{supp}(f*g)^c$.

$$\boxed{\int_{-\infty}^{\infty} f(x-y)g(y)\,dy} = 0$$

every single this thing will be zero.

Let $g(y_0) \neq 0$ for some $-\infty < y_0 < \infty$.

If $f(x-y_0) \neq 0$, then $x - y_0 \in \operatorname{supp} f$.

$g(y_0) \neq 0 \Rightarrow y_0 \in \operatorname{supp} g$.

$\Rightarrow x - y_0 + y_0 \in \operatorname{supp} f + \operatorname{supp} g$.

$\Rightarrow x \in \operatorname{supp} f + \operatorname{supp} g$

but if $x \in B(x_0, \varepsilon) \Rightarrow$ this is contradiction

$\Rightarrow f(x-y_0) = 0$ for any $x \in B(x_0, \varepsilon)$.

$\Rightarrow f(x-y_0)g(y_0) = 0$ for any $x \in B(x_0,\varepsilon)$.

$\Rightarrow f*g\ (x)$ is zero

$\Rightarrow$ On $B(x_0,\varepsilon)$, $f*g$ vanishes

$\Rightarrow x_0 \in \left(\operatorname{supp}(f*g)\right)^c$.

4. $u_t = k u_{xx}$ $\quad -\infty < x < \infty, \quad t > 0.$

$$\boxed{u(x,0) = \frac{1}{2}\big(H(x+1) - H(1-x)\big)}$$

$H(x)$: heavyside ftn.

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases}$$

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t)\, u(y,0)\, dy.$$

$$\left(\Phi(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}} \right.$$

$$= \frac{1}{2\sqrt{4\pi k t}} \int_{\boxed{-\infty}}^{\boxed{\infty}} e^{-\frac{(x-y)^2}{4kt}} \cdot \big(H(y+1) - H(1-y)\big)\, dy$$

$$H(y+1) - H(1-y) = \begin{cases} +1 & y > 1 \\ 0 & \text{o.w.} \\ -1 & y < -1. \end{cases}$$

$$= {}'' \int_{1}^{\infty} \quad dy - \int_{-\infty}^{-1} \quad dy$$

$$y = -z + 2x$$

$$= \boxed{} \int_{1}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\,dy - \int_{-\infty}^{-1} e^{-\frac{(x-y)^2}{4kt}}\,dy.$$

(Think about $y = 5x \longleftrightarrow y = -3x$)

$$(-4x)^2 = (4x)^2 = (x-(-3x))^2$$

$$y = -z + 2x$$

$$5x = -(-3x) + 2x$$

$$\int_{\infty}^{2x+1} e^{-\frac{(z-x)^2}{4kt}}\,d(-z) \qquad (z-x)^2 = (x-z)^2$$

$$\int_1^{\infty} f - \int_{2x+1}^{\infty} f = \int_1^{2x+1} f \qquad = \int_{2x+1}^{\infty} e^{-\frac{(x-z)^2}{4kt}}\,dz.$$

$$= \boxed{} \int_{1}^{2x+1} e^{-\frac{(x-y)^2}{4kt}}\,dy \qquad (x-y)^2 = (y-x)^2$$

$$y - x \xrightarrow[+x]{} y$$

$$\frac{1}{2\sqrt{4\pi kt}}$$

$$= \boxed{} \int_{-x+1}^{x+1} e^{-\frac{y^2}{4kt}}\,dy = \int_0^{x+1} \boxed{e^{-\frac{y^2}{4kt}}}\,dy - \int_0^{-x+1} ''$$

change of coord

$$= \mathrm{Erf}(\;) \quad e^{-\text{coeff.}\,y^2}\,dy \quad \mathrm{Erf}(\;)$$

$$\mathrm{Erf}(x) = \boxed{\frac{2}{\sqrt{\pi}}} \int_0^{\boxed{x}} e^{\boxed{-y^2}}\, dy$$

$$\int_0^x e^{-\frac{y^2}{4kt}}\, dy = \int_0^{\frac{x}{\sqrt{4kt}}} e^{-z^2}\, d(\sqrt{4kt}\, z)$$

$$(y = \sqrt{4kt}\, z)$$

$$= \sqrt{4kt} \int_0^{\boxed{}} e^{-z^2}\, dz$$

$$= \sqrt{4kt} \cdot \frac{\sqrt{\pi}}{2}\, \mathrm{Erf}\left(\frac{x}{\sqrt{4kt}}\right).$$