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This exam has 10 pages, 4 questions, and a total of **100** points.

1. Assume $u : \mathbf{R}^2 \to \mathbf{R}$ is C^2.

   (a) (10 points) Solve

$$u_{tt} = u_{xx} \text{ for } x \in \mathbf{R} \text{ and } t > 0 \text{ where } \begin{cases} u(x,0) = e^{-x}, & x \in \mathbf{R} \\ u_t(x,0) = 0, & x \in \mathbf{R}. \end{cases}$$

$\phi(x) = e^{-x}$

$\psi(x) = 0$

By d'Alembert's formula

$$u(x,t) = \tfrac{1}{2}\phi(x+t) + \tfrac{1}{2}\phi(x-t) + \int_{x-t}^{x+t} 0\,dy$$

$$= \frac{1}{2} e^{-x-t} + \frac{1}{2} e^{-x+t}$$

for $x \in \mathbb{R}$, $t > 0$.

   (b) (10 points) Solve

$$u_{tt} = u_{xx}, \text{ for } x > 0 \text{ and } t > 0 \begin{cases} u(0,t) = 0, & t > 0 \\ u(x,0) = e^{-x}, & x > 0 \\ u_t(x,0) = 0, & x > 0. \end{cases}$$

$\phi(x) = e^{-x}$

$\psi(x) = 0$

By d'Alembert's formula

$$u(x,t) = \tfrac{1}{2}\phi(x+t) - \tfrac{1}{2}\phi(t-x) + \int_{t-x}^{x+t} 0\,dy$$

$$= \frac{1}{2} e^{-x-t} - \frac{1}{2} e^{-t+x} \text{ for } x \in (0,t)$$

and

$$u(x,t) = \tfrac{1}{2}\phi(x+t) + \tfrac{1}{2}\phi(x-t) + \int_{x-t}^{x+t} 0\,dy$$

$$= \frac{1}{2} e^{-x-t} + \frac{1}{2} e^{-x+t} \text{ for } x \in (t,\infty).$$

(c) (15 points) Suppose $u : \mathbf{R}^2 \to \mathbf{R}$ is C^2. Find the general solution of

$$u_{tt} + u_{xt} - 12u_{xx} = 0$$

and show it satisfies the PDE.

Here

$$\mathcal{L}\{u\} = \partial_{tt}u + \partial_{tx}u - 12\partial_{xx}u$$

and by Clairaut's Thm,

$$= \partial_{tt}u + 4\partial_{xt}u - 3\partial_{tx}u - 12\partial_{xx}u$$

$$= (\partial_t + 4\partial_x)(\partial_t - 3\partial_x)u = 0$$

Then

$$(\partial_t + 4\partial_x)u = 0 \quad \text{or} \quad (\partial_t - 3\partial_x)u = 0$$

are transport equations where

$$f(x-4t) \quad \text{and} \quad g(x+3t)$$

form the general solution.

Thus

$$u(x,t) = f(x-4t) + g(x+3t)$$

is the general solution where $f, g \in C^2(\mathbb{R})$.

Note:

$$\begin{array}{r|l}
u_{tt} & 16f''(x+4t) + 9g''(x-3t) \\
+u_{xt} & -4f''(x+4t) + 3g''(x-3t) \\
-12u_{xx} & -12f''(x+4t) - 12g''(x-3t) \\
\hline
 & 0
\end{array}$$

2. Consider the equation

$$(y+u)u_x + yu_y = x - y$$

where $u : \mathbf{R}^2 \to \mathbf{R}$ is C^1.

(a) (10 points) Given the initial data

$$u(x,1) = 1 + x$$

find the characteristics.

Our system of ODEs is

$$x_\tau = y + u \qquad x(s,0) = s$$
$$y_\tau = y \qquad y(s,0) = 1$$
$$u_\tau = x - y \qquad u(s,0) = 1+s$$

Now

$$x_{\tau\tau} = y_\tau + u_\tau = y + x - y = x$$

and so

$$x(s,\tau) = a(s)e^{\tau} + b(s)e^{-\tau}$$

$\Rightarrow$

$$x(s,0) = a + b = s$$

$x_\tau(s,0) = y(s,0) + u(s,0)$, from the system

$= 2 + s$

$= a - b$, from the general sol'n

$\Rightarrow$ $a = 1+s$, $b = -1$

$\Rightarrow$ $x(s,\tau) = (1+s)e^{\tau} - e^{-\tau}$

Next,

$$y_\tau = y \Rightarrow y(s,\tau) = e^{\tau}$$

and our characteristic curves are

$$x = (1+s)e^{\tau} - \frac{1}{e^{\tau}}$$
$$= (1+s)y - 1/y.$$

(b) (10 points) Given the initial data

$$u(x,1) = 1 + x$$

find the explicit solution for u or explain why none can exist.

From above,

$$u_\tau = se^{\tau} - e^{-\tau} \Rightarrow u(s,\tau) = se^{\tau} + e^{-\tau}$$

Solving for s and τ yields

$$x = (1+s)y - 1/y \Rightarrow s = \frac{x + 1/y}{y} - 1 = x/y + 1/y^2 - 1$$

and

$$y = \ln(\tau).$$

Hence

$$u(x,y) = \left(x/y + 1/y^2 - 1\right)y + 1/y$$

$$= x - y + \frac{2}{y}.$$

(c) (5 points) Give an example of a connected curve C in $\mathbf{R}^2$ such that the PDE with prescribed data on that curve cannot be solved.

Choose C to be a characteristic, let $s = 0$,

$$\{(x,y) \mid x = (1+0)y - 1/y\}$$

or a curve that never intersects a characteristic.

3. (20 points) Show that for

$$u = f\left(\frac{x}{t}\right)$$

to be a nonconstant solution of

$$u_t + a(u)u_x = 0$$

then f must be the inverse function of a.

Idea : Plug in $u(x,t) = f\left(\frac{x}{t}\right)$ into the equation and use Chain Rule appropriately.

$u_x(x,t) = \frac{\partial}{\partial x}\left(f\left(\frac{x}{t}\right)\right) = \frac{1}{t} \cdot f'\left(\frac{x}{t}\right)$ (Chain Rule).

$u_t(x,t) = \frac{\partial}{\partial t}\left(f\left(\frac{x}{t}\right)\right) = -\frac{x}{t^2} \cdot f'\left(\frac{x}{t}\right)$ (").

Now, if you plug them into the given equation $u_t + a(u)u_x = 0$, you get

$$-\frac{x}{t^2} f'\left(\frac{x}{t}\right) + a\left(f\left(\frac{x}{t}\right)\right) \cdot \frac{1}{t} f'\left(\frac{x}{t}\right) = 0.$$

We can rewrite this as

$$-\left(\frac{x}{t} - a\left(f\left(\frac{x}{t}\right)\right)\right) \cdot \frac{1}{t} f'\left(\frac{x}{t}\right) = 0.$$

So, there are two cases : $f'\left(\frac{x}{t}\right) = 0$ OR $a\left(f\left(\frac{x}{t}\right)\right) = \frac{x}{t}$. However, this holds for any real numbers x & t. So, we can see what happens at $x = \lambda t$ where λ is your favorite number. Now, for any $\lambda \in \mathbb{R}$, we should have

$$f'(\lambda) = 0 \quad \text{OR} \quad a(f(\lambda)) = \lambda.$$

But, in fact, if the product of two continuous functions is 0, then one of them should be the zero function.

In this problem, two functions are $f'(\lambda)$ & $a(f(\lambda)) - \lambda$. But, we know that u is not constant which implies that $f'(\lambda)$ is not constantly 0. Hence, $a(f(\lambda)) - \lambda \equiv 0$.

Therefore, f is the inverse function of a.

ϟ It is necessary to check that $a(f(\lambda)) = \lambda$ for all $\lambda \in \mathbb{R}$.

4. In lecture we proved the following:

Theorem: Suppose that u_1 and u_2 solve the IVP $u_{tt} = c^2 u_{xx}$ with displacement functions ϕ_1 and ϕ_2 and velocity functions ψ_1 and ψ_2 respectively.

Then

$$\|\phi_1 - \phi_2\|_\infty < \epsilon \quad \text{and} \quad \|\psi_1 - \psi_2\|_\infty < \epsilon \Rightarrow \|u_1 - u_2\|_\infty < (1+t)\epsilon$$

for $t > 0$.

(a) (5 points) State a similar result for the PDE

$$u_{tt} = c^2 u_{xx} + f(x,t).$$

Theorem (Continuous Dependence on Data)
Suppose that u_1 and u_2 solve the IVP

$$u_{tt} = c^2 u_{xx} + f(x,t)$$

with displacement functions ϕ_1 and ϕ_2, velocity functions ψ_1 and ψ_2, and forcing terms f_1 and f_2 respectively.

If

$$\|\phi_1 - \phi_2\|_\infty, \|\psi_1 - \psi_2\|_\infty, \text{ and } \|f_1 - f_2\|_\infty$$

are bounded by ϵ then

$$\|u_1 - u_2\|_\infty < \left(1 + t + \frac{t^2}{2}\right)\epsilon$$

for $t > 0$.

This is $\frac{1}{2}t^2$ not t^2

if you did not make a mistake on writing the Duhamel solution.

(b) (15 points) Prove your result for (a) for $c = 1$.

Idea: Use Duhamel solution and simply bound the difference using maximum values.

$$u_i(x,t) = \frac{1}{2}\left[\phi_i(x+ct) + \phi_i(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi_i(y)dy + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f_i(y,s)dyds$$

is the solution for each equation $(i=1,2)$.

By subtracting u_2 from u_1, we get

$$u_1(x,t) - u_2(x,t) = \frac{1}{2}\left[\phi_1(x+ct)+\phi_1(x-ct)\right] - \frac{1}{2}\left[\phi_2(x+ct)+\phi_2(x-ct)\right]$$
$$+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi_1(y)dy - \frac{1}{2c}\int_{x-ct}^{x+ct}\psi_2(y)dy$$
$$+\frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f_1(y,s)dyds - \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f_2(y,s)dyds$$

Hence, $$|u_1(x,t) - u_2(x,t)| \le \frac{1}{2}\left|\phi_1(x+ct) - \phi_2(x+ct)\right| + \frac{1}{2}\left|\phi_1(x-ct) - \phi_2(x-ct)\right|$$
$$+\frac{1}{2c}\int_{x-ct}^{x+ct}|\psi_1(y)-\psi_2(y)|dy + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)}|f_1(y,s)-f_2(y,s)|dyds$$

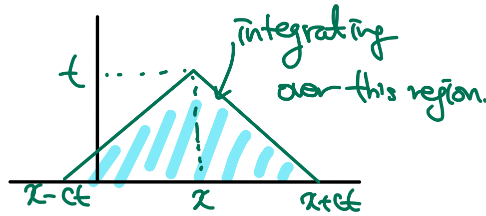
Here, we used the triangle inequality and its integral version, namely, $\left|\int_a^b f\right| \le \int_a^b |f|$.

Recalling the definition $\|\cdot\|_\infty = \sup|\cdot|$, we get

$$\frac{1}{2}\left|\phi_1(x+ct) - \phi_2(x+ct)\right| \le \frac{1}{2}\|\phi_1-\phi_2\|_\infty < \frac{1}{2}\varepsilon$$
$$\frac{1}{2}\left|\phi_1(x-ct) - \phi_2(x-ct)\right| \le \frac{1}{2}\|\phi_1-\phi_2\|_\infty < \frac{1}{2}\varepsilon$$
$$\frac{1}{2c}\int_{x-ct}^{x+ct}|\psi_1(y)-\psi_2(y)|dy \le \frac{1}{2c}\int_{x-ct}^{x+ct}\|\psi_1-\psi_2\|\,dy < \frac{1}{2c}\cdot 2ct\cdot\varepsilon$$

and finally

$$\frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)}|f_1(y,s)-f_2(y,s)|dyds \le \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)}\|f_1-f_2\|_\infty dyds$$
$$< \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)}\varepsilon\, dyds$$
$$= \frac{1}{2c}\int_0^t 2c(t-s)\varepsilon\, ds = \frac{1}{2c}ct^2\varepsilon = \frac{1}{2}t^2\varepsilon.$$



Adding them up, we obtain that

$$|u_1(x,t) - u_2(x,t)| < \varepsilon + t\varepsilon + \frac{1}{2}t^2\varepsilon = \left(1+t+\frac{t^2}{2}\right)\varepsilon \quad \text{for any } x\in\mathbb{R}.$$

Therefore, $$\|u_1-u_2\|_\infty := \sup_{x\in\mathbb{R}}|u_1(x,t)-u_2(x,t)| < \left(1+t+\frac{t^2}{2}\right)\varepsilon.$$

⚡ To avoid the issue of getting $\le$ instead of $<$, you need to be a bit careful.