Homework 7 - Spring 2020 MATH 126-001 - Introduction to PDEs

1. Let $U \subset \mathbf{C}$ be an open domain and $f : U \to \mathbf{C}$.

   Suppose that

   $$f(x+iy) = u(x,y) + iv(x,y) \quad \text{and} \quad \overline{f}(x+iy) = u(x,y) - iv(x,y)$$

   are holomorphic on U.

   Find f.

2. Let
   $$u(x,y) = \frac{x}{2} - 6x^2 + 4y - 6x^2y + 6y^2 + 2y^3.$$

   (a) Find all harmonic conjugates of u.

   (b) If $z = a + ib \in \mathbf{C}$ then the **real** and **imaginary** components of z are defined by
   $$\mathrm{Re}(z) = a \quad \text{and} \quad \mathrm{Im}(z) = b.$$
   Find a function $f : \mathbf{C} \to \mathbf{C}$ in terms of $z \in \mathbf{C}$ such that
   $$\mathrm{Re}(f) = u.$$

   (c) Find the largest domain in $\mathbf{C}$ that the function f from (b) is holomorphic on.

3. (a) Find
   $$\oint_{|z-3|=2} \frac{e^{-z^2}}{z^3 - 9z^2 + 11z + 21}\, dz.$$

   (b) Find
   $$\oint_{|z-1|=2} \frac{\sin(z)}{z^2 - 4}\, dz.$$

4. Suppose the function $f : \mathbf{C} \to \mathbf{C}$ is holomorphic on
   $$A = \{z \in \mathbf{C} \mid 2 \le |z| \le 3\}.$$
   Furthermore,
   $$|f(z)| \le 16 \text{ on } |z| = 2 \text{ and } |f(z)| \le 36 \text{ on } |z| = 3.$$
   Show that
   $$|f(z)| \le 4|z|^2$$
   on A.

5. This problem outlines a "bar room" / informal proof of Cauchy's Integral Formula.

   Assume U is a simply connected domain. Let f be holomorphic on ∂U and inside U and suppose $z_0 \in U$. We know

   $$\oint_{\partial U} \frac{f(z)}{z - z_0} dz = \oint_{C_r} \frac{f(z)}{z - z_0} dz$$

   where C is a circle centered at z_0 with radius r.

   (a) Express $z = z_0 + re^{it}$ where r is given by C and $t \in [0, 2\pi]$ and rewrite the above integral in polar form.

   (b) From (a) let $r \to 0$ in the integrand. Then integrate and find

   $$\oint_{\partial\Omega} \frac{f(z)}{z - z_0} dz.$$

   (c) What lacked rigor with what you did in part (5b)?

6. Find all radially symmetric solutions of

   $$u_{xx} + u_{yy} + u_{zz} = k^2 u.$$

7. Find all radially symmetric solutions of

   $$u_{xx} + u_{yy} = k^2 u.$$

8. Determine if the maximum principle for harmonic functions applies to the function

   $$u(x, y) = \frac{1 - x^2 - y^2}{1 - 2x + x^2 + y^2}$$

   over the disk

   $$D = \{x \in \mathbf{R}^2 \mid |\mathbf{x}| \leq 1\}.$$

9. Solve

   $$u_{xx} + u_{yy} = 0$$

   on the set

   $$D = \{x \in \mathbf{R}^2 \mid |\mathbf{x}| \leq 1\}$$

   where

   $$u = 1 + 3\sin\theta \text{ on } \partial D.$$

   (Here θ denotes the polar angle on the boundary of D)

Homework 7 Solution

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1. Idea: Use Cauchy-Riemann equations to check holomorphicity.

f is holomorphic $\Rightarrow$ $u_x = v_y$ & $u_y = -v_x$ on $\mathcal{U}$.

$\bar{f}$ is holomorphic $\Rightarrow$ $u_x = (-v)_y$ & $u_y = -(-v)_x = v_x$

Hence, $v_y = u_x = -v_y \Rightarrow v_y \equiv 0$, $u_x \equiv 0$. Similarly, $u_y \equiv 0$, $v_x \equiv 0$.

Therefore, $u(x,y)$ and $v(x,y)$ should be constant. In particular, $f \equiv c$ for some $c \in \mathbb{C}$.

2. Idea: Harmonic conjugate is the imaginary part of the holomorphic function having u as the real part.

(a) By the Cauchy-Riemann equations, we have v (the harmonic conjugate of u) satisfy $v_y = u_x = \frac{1}{2} - 12x - 12xy$ & $v_x = -u_y = -4 + 6x^2 - 12y - 6y^2$.

Therefore, $v(x,y) = \frac{1}{2}y - 12xy - 6xy^2 + f(x)$ and $f'(x) = -4 + 6x^2$, so we get $v(x,y) = \frac{1}{2}y - 12xy - 6xy^2 - 4x + 2x^3 + C$, but we want $v(0,0) = 0$.

Hence, $C = 0$.

(b) $f(x+iy) = u(x,y) + iv(x,y)$

$$= \frac{x}{2} - 6x^2 + 4y - 6x^2y + 6y^2 + 2y^3 + i\left(\frac{1}{2}y - 12xy - 6xy^2 - 4x + 2x^3\right)$$

$$\left[(x+iy)^3 = x^3 + 6x^2yi - 6xy^2 - y^3 \cdot i\right] \times 2i$$

$$\left[(x+iy)^2 = x^2 + 2xyi - y^2\right] \times -6$$

$$\left[(x+iy)^1 = x + iy\right] \times \left(\frac{1}{2} - 4i\right)$$

Therefore, $f(z) = 2iz^3 - 6z^2 + \left(\frac{1}{2} - 4i\right)z$.

(c) $f(z)$ is a polynomial of z, so it is defined all over $\mathbb{C}$ and holomorphic everywhere. The largest domain is $\mathbb{C}$.

3. Idea: Use Cauchy's Integral Formula after checking that the integrand is holomorphic.

(a) Cauchy's integral formula tells us that if $f(z)$ is holomorphic in $\mathcal{U}$, then

$\oint_{\partial D} \frac{f(z)}{z-a} dz = 2\pi i \cdot f(a)$ for any $a \in D$ where D is a closed disk in $\mathcal{U}$.

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We can easily see that $z^3-9z^2+11z+21 = (z-3)(z^2-6z-7)$
$= (z-3)(z-7)(z+1)$.

Let $f(z)=\dfrac{e^{-z^2}}{(z+1)(z-7)}$. As e^{-z^2}, $\frac{1}{z+1}$, $\frac{1}{z-7}$ are holomorphic outside $z=-1$ & $z=7$, we can choose $U=\{z\in\mathbb{C}: |z-3|<3\}$ over which f is holomorphic. Let $D=\{z\in\mathbb{C}: |z-3|\le 2\}$. Then, we can apply Cauchy's Integral formula:

$$\oint_{|z-3|=2}\frac{e^{-z^2}}{z^3-9z^2+11z+21} = \oint_{|z-3|=2}\frac{f(z)}{z-3}dz = 2\pi i\cdot f(3) = 2\pi i\cdot\frac{e^{-3^2}}{(3+1)(3-7)} = -\frac{1}{8e^9}\cdot\pi i.$$

(b) Similarly, let $f(z)=\dfrac{\sin(z)}{z+2}$ and $U=\{z\in\mathbb{C}: |z-1|<3\}$ and $D=\{z\in\mathbb{C}: |z-1|\le 2\}$. Then, as $z=2$ belongs to D, we have the following formula:

$$\oint_{|z-1|=2}\frac{\sin(z)}{z^2-4}dz = \oint_{|z-1|=2}\frac{f(z)}{z-2}dz = 2\pi i\cdot f(2) = \frac{1}{2}\sin(2)\cdot\pi i.$$

4. Idea: Use the maximum modulus principle after checking the assumptions.

The maximum modulus principle can be applied to $f(z)/z^2$ which is holomorphic in the connected open subset $\{z\in\mathbb{C}: 2<|z|<3\}$. The conditions given are $|f(z)/z^2|\le 4$ for $|z|=2$ & $|f(z)/z^2|\le 4$ for $|z|=3$. This implies that on the boundary of the open subset, the absolute value is bounded above by 4. By the maximum modulus principle, $|f(z)/z^2|\le 4$ on the open subset. It is equivalent to saying that $|f(z)|\le 4|z^2|$ inside & on the boundary.

Maximum modulus principle is a holomorphic function version of the maximum principle for a harmonic function.

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5. Idea: Follow the instruction.

(a) $\oint_{C_r} \frac{f(z)}{z-z_0}dz = \int_0^{2\pi} \frac{f(z_0+r\cdot e^{it})}{r\cdot e^{it}} r\cdot i\cdot e^{it}dt$ $\left(dz = \frac{d(z_0+r\cdot e^{it})}{dt}dt = r\cdot i\cdot e^{it}dt\right.$

$= \int_0^{2\pi} f(z_0+r\cdot e^{it})\cdot i\, dt$

(b) Only thing affected by the change of r is $f(z_0+r\cdot e^{it})$.

As r goes to 0, it goes to $f(z_0)$. The integral does not depend on r and as f is continuous, $\oint_{\partial\Omega} \frac{f(z)}{z-z_0}dz = \oint_{C_r}\frac{f(z)}{z-z_0}dz = \lim_{r\to 0}\oint_{C_r}\frac{f(z)}{z-z_0}dz$

$= \int_0^{2\pi} f(z_0)\cdot i\,dt$

$= f(z_0)\cdot i\cdot\int_0^{2\pi} 1\,dt = 2\pi i\cdot f(z_0)$.

(c) In the proof of part b, we used the fact that

$$\lim \oint = \oint \lim.$$

In order to have a concrete proof, we need to check under our assumption if the above "commutativity" holds.

6. Idea: Use the spherical coordinate Laplacian.

In spherical coordinates, the Laplacian can be written as $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{\partial f}{\partial r}\right)$ + other terms where the "other terms" are partial derivatives w.r.t the angles. However, we are looking for the solutions which are radially symmetric. So, the equation can be written as

$$\frac{1}{r^2}\cdot(r^2\cdot u_r)_r = k^2 u$$

$$\| $$

$$u_{rr} + \frac{2}{r}u_r.$$

However, in fact, the left hand side can be written as $\frac{1}{r}(ru)_{rr}$. Therefore, the equation now becomes $\frac{1}{r}(ru)_{rr} = k^2 u$ and $(ru)_{rr} = k^2\cdot ru$. We already know that $f'' - k^2 f = 0$ has the solution $f(x) = C_1 e^{kx} + C_2 e^{-kx}$. $\therefore ru(r) = C_1 e^{kr} + C_2 e^{-kr}$.

We get $u(r) = C_1\cdot\frac{e^{kr}}{r} + C_2\cdot\frac{e^{-kr}}{r}$.

↯ Changing $u_{rr} + \frac{2}{r}u_r$ into $\frac{1}{r}(ru)_{rr}$ is crucial but a little bit tricky.

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7. Idea: Use the spherical coordinate Laplacian.

As the equation is 2-dim'l setup, we have $u_{xx}+u_{yy}=u_{rr}+\frac{1}{r}u_r$ given that u is radially symmetric. So, the equation now becomes:

$$u_{rr}+\frac{1}{r}u_r=k^2u.$$

In other words, $r^2u''+ru'-k^2\cdot r^2u=0$. However, this looks similar to the Bessel's equation. We can try to change the coordinates to make this be the Bessel's equation. Let c be a scalar and $v(r)=u(cr)$. Then,

$v'(r)=cu'(cr)$ and $v''(r)=c^2\cdot u''(cr)$. If we plug in cr to the original equation, we get $\underbrace{c^2r^2u''(cr)}_{=r^2v''(r)}+\underbrace{cr\,u'(cr)}_{=r\,v'(r)}-\underbrace{k^2c^2r^2u(cr)}_{=k^2c^2r^2v(r)}=0$.

The Bessel's equation has the coefficient of this be $+r^2$, so we can guess that $c^2=-\frac{1}{k^2}$ or $c=\frac{1}{ki}$. Then, $v(r)$ becomes a solution of the Bessel's equation of order 0. $\therefore u(r)=v(\frac{r}{c})=v(kir)=C_1J_0(kir)+C_2Y_0(kir)$ where J_0: the solution of the first kind, Y_0: that of the second kind.

8. Idea: Express the numerator and the denominator in terms of $z=x+iy$ and $\bar{z}=x-iy$.

The denominator is $|z-1|^2=(z-1)\overline{(z-1)}=(z-1)(\bar{z}-1)$ and the numerator is $1-|z|^2=1-z\bar{z}$. One way to prove that u is harmonic is to find a holomorphic function $f(z)$ which satisfies $f(z)+\overline{f(z)}=2u$. From the above observation, we have a guess $f(z)=\frac{1}{z-1}$. In this case, we get $f(z)+\overline{f(z)}=\frac{z+\bar{z}-2}{(z-1)(\bar{z}-1)}$. We now observe that the numerator and the denominator can be "assembled" to generate what we are looking for. A careful consideration suggests $f(z)=\frac{-2}{z-1}-1=\frac{1+z}{1-z}$. It is holomorphic on $\{z\in\mathbb{C}: |z|<1\}$. Therefore, $u=\mathrm{Re}(f(z))$ is harmonic on $\{(x,y)\in\mathbb{R}^2: x^2+y^2<1\}=U$. Now, we can apply the maximum principle if u is continuous along the boundary $x^2+y^2=1$. However, along the boundary, the denominator is $1-2x+1=2(1-x)$ becomes 0 at $x=1$. So, it is not just discontinuous, but it is not defined. So, you cannot obtain the maximum.

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9. Idea: Laplace's equation on spherical domain (8.4.2 in Shearer & Levy)

Using Separation of Variables $u(r,\theta) = R(r)H(\theta)$, we have the candidate for u as follows: $u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n \cdot (A_n \cos n\theta + B_n \sin n\theta)$ where A_n and B_n are $\frac{1}{\pi} \cdot \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta$ and $\frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$ for the boundary condition $u = f$ on ∂D.

$A_0 = \frac{1}{\pi} \int_0^{2\pi} (1 + 3\sin\theta) \cdot \cos 0 \, d\theta = \frac{1}{\pi} \int_0^{2\pi} (1 + 3\sin\theta) d\theta = \frac{1}{\pi} \cdot 2\pi = 2.$

$A_n = \frac{1}{\pi} \int_0^{2\pi} (1 + 3\sin\theta) \cos n\theta \, d\theta = \frac{1}{\pi} \int_0^{2\pi} (\cos n\theta + 3\sin\theta \cos n\theta) d\theta = 0 + 0 = 0,$

$B_n = \frac{1}{\pi} \int_0^{2\pi} (1 + 3\sin\theta) \cdot \sin n\theta \, d\theta = \frac{1}{\pi} \int_0^{2\pi} (\sin n\theta + 3\sin\theta \cdot \sin n\theta) d\theta = 0 + 0 \quad (n \neq 1)$

OR

$0 + \frac{3}{\pi} \int_0^{2\pi} \sin^2\theta \, d\theta \; (n = 1)$

$\int_0^{2\pi} \sin^2\theta \, d\theta = \int_0^{2\pi} \frac{1 - \cos 2\theta}{2} d\theta = \frac{1}{2} \cdot 2\pi.$ So, $B_1 = \frac{3}{\pi} \cdot \pi = 3.$

$\therefore u(r,\theta) = 1 + r \cdot 3 \cdot \sin\theta = 1 + 3r\sin\theta.$

↳ $1 + 3r\sin\theta$ is simply $1 + 3y$ in the standard coordinate. Its Laplacian is zero as both partial derivatives vanish.