Homework 6 - Spring 2020 MATH 126-001 - Introduction to PDEs

1. Let $E \subseteq \mathbf{R}$ and define a sequence of functions $f_n : E \to \mathbf{R}$. Prove that $f_n \to f$ uniformly on E if and only if
$$\|f_n - f\|_\infty \to 0$$
as $n \to \infty$.

2. Let
$$f_n(x) = x^2 e^{-nx}$$
be a sequence of functions.
   (a) Does the sequence (f_n) converge pointwise on $[0, \infty)$? Prove your claim.
   (b) Does the sequence (f_n) converge uniformly on $[0, \infty)$? Prove your claim.

3. Let
$$f_n(x) = \frac{1}{1 + n^2 x^2}$$
form a sequence of functions on $|x| < 1$.
   (a) Show
$$\lim_{n\to\infty} \lim_{x\to 0} f_n(x) \neq \lim_{x\to 0} \lim_{n\to\infty} f_n(x).$$
   (b) Why does the sequence not converge uniformly on $|x| < 1$?

4. Use a trigonometric series expansion of x on $[-\pi, \pi]$ to find the value of
$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

5. Let T_n be a trigonometric polynomial on $[-\pi, \pi]$ of degree n. Find
$$\|T_n\|_2.$$

6. Let f be a 2π-periodic, continuous function where
$$\sum |a_k| \quad \text{and} \quad \sum |b_k|$$
both converge. Show the Fourier series of f converges absolutely and uniformly to f.

7. Let $f : \mathbf{R} \to \mathbf{R}$ where for any $\epsilon > 0$ there is a trigonometric polynomial T such that
$$|f(x) - T(x)| < \epsilon$$
for all $x \in \mathbf{R}$.

(a) Show f is the uniform limit of trigonometric polynomials.

(b) Show f is 2π-periodic and continuous.

8. Solve the initial boundary value problem

$$\begin{aligned} u_t &= 4u_{xx}, && 0 < x < \pi,\ t > 0, \\ u(0,t) &= 0 = u(\pi,t), && t > 0, \\ u(x,0) &= \sin x - 3\sin 5x, && 0 < x < \pi. \end{aligned}$$

9. Solve the initial boundary value problem

$$\begin{aligned} u_{tt} &= 9u_{xx}, && 0 < x < 1,\ t > 0, \\ u(0,t) &= 0 = u(1,t), && t > 0, \\ u(x,0) &= 2\sin(\pi x) + 7\sin(3\pi x), && 0 < x < 1, \\ u_t(x,0) &= 2\sin(\pi x), && 0 < x < 1. \end{aligned}$$

Homework 6 Solution

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1. Idea: $\|\cdot\|_\infty$ the infinite norm is the most powerful. So, one of "if and only if" should be easy.

Suppose that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. This means that $\sup_{x\in E} |f_n(x) - f(x)| \to 0$ as $n\to\infty$.
So, $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ $\sup_{x\in E} |f_n(x) - f(x)| < \varepsilon$. But $\sup_{x\in E} |f_n(x) - f(x)| \geq |f_n(x) - f(x)|$ for all $x \in E$. In particular, this means $f_n \to f$ uniformly.

The reverse direction: Suppose $f_n \to f$ uniformly, then $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|f_n(x) - f(x)| < \varepsilon$ for any x. However, if $g(x) < \varepsilon$ for all $x \in E$, this implies that $\sup_{x\in E} g(x) \leq \varepsilon$ (in general). If we consider $g = f_n - f$ in this situation, $\sup_{x\in E} |f_n(x) - f(x)| \leq \varepsilon$. Therefore, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\|f_n - f\| \leq \varepsilon$ so that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

2. Idea: Just check following the definitions of convergence.

(a) Fix $x \in [0,\infty)$. $f_n(x) = \frac{x^2}{e^{nx}}$. If $x=0$, this is zero. However, if $x>0$, then e^{nx} grows (literally) exponentially, so the denominator will go to ∞ so that $f_n(x) \to 0$. Therefore, $f_n(x)$ converges pointwise on $[0,\infty)$.

(b) Using Problem 1, we can check if it converges uniformly by considering $\|f_n - f\|_\infty$. If the sequence converges uniformly, then f should be the zero function by part a. Now, what is $\|f_n - 0\|$? It is the supremum of $\left|\frac{x^2}{e^{nx}}\right|$ over $[0,\infty)$.

What we need to do now is to compute a maximum of $f_n(x) = x^2 \cdot e^{-nx}$.
$f_n'(x) = 2x\cdot e^{-nx} - nx^2 \cdot e^{-nx} = x\cdot e^{-nx}(2 - nx)$. $f_n\left(\frac{2}{n}\right) = \left(\frac{2}{n}\right)^2 \cdot e^{-2} = \frac{1}{n^2}\cdot\left(\frac{2}{e}\right)^2$.
$f_n'(x)$ is positive for $x \in [0, \frac{2}{n})$ and negative for $x \in (\frac{2}{n}, \infty)$. Therefore, $\|f_n - f\|_\infty = \left(\frac{2}{e}\right)^2 \cdot \frac{1}{n^2}$ and it goes to zero. Finally, by Problem 1, they uniformly converge to 0.

3. Idea: Just do it!

(a) $f_n(x)$ is continuous $\Rightarrow$ $\lim_{x\to 0} f_n(x) = f_n(0) = 1$. So, the left hand side is 1.
As $n\to\infty$, $n^2x^2 \to \infty$ unless $x = 0$. Hence, $\lim_{n\to\infty} f_n(x) = 0$ if $x \neq 0$. The limit $x \to 0$ is taken outside of $x=0$, so the right hand side is 0. $1 \neq 0$.

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(b) If the sequence converges uniformly on $|x|<1$, the function f to which f_n converges should be continuous. ($f_n \to f$ uniformly on $[a,b]$, then f is continuous. Note that $[a,b]$'s closedness does not really matter because we can choose $[-\frac{1}{2}, \frac{1}{2}]$ in this problem instead of $(-1,1)$.)
So, $f(x)$ should be continuous. Especially at $x=0$. However, $f(0) = \lim_{n\to\infty} f_n(0) = \lim_{n\to\infty}\lim_{x\to 0} f_n(x)$ and $\lim_{x\to 0} f(x) = \lim_{x\to 0}\lim_{n\to\infty} f_n(x)$ and we checked in part a that they are different. So, it is a contradiction and the sequence does not converge uniformly.

4. Idea: Noting that $f(x)=x$ on $[-\pi,\pi]$ is an odd function, we know that the Fourier sine series should be the same as the Fourier series, but the Fourier cosine series is different. So, we should try two trigonometric series.

It turns out that the Fourier series is not working. It does not have terms w/ n^2.
So, we take the formula for the Fourier cosine series, or equivalently consider the Fourier series of the even function $\begin{cases} -x & \text{on } [-\pi,0] \\ x & \text{on } [0,\pi]. \end{cases}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx \qquad \int uv' = uv - \int u'v.$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\cdot\frac{1}{2}x^2\Big|_0^{\pi} = \pi.$$

$$= \frac{2}{\pi}\left(x\,\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi} 1\cdot\frac{\sin nx}{n}\,dx\right)$$

$$= \frac{2}{\pi}\left(\pi\cdot\frac{0}{n} - 0 - 1\cdot\frac{\cos nx}{-n^2}\Big|_0^{\pi}\right)$$

$$= \frac{1}{\pi}\frac{2}{n^2}\left((-1)^n - 1\right) = \frac{-4}{\pi n^2} \text{ if } n \text{ is odd and } 0 \text{ o.w.}$$

Hence, the Fourier cosine expansion of $f(x)=x$ defined over $[0,\pi]$ is $\frac{\pi}{2} + \frac{1}{\pi}\sum_{m=0}^{\infty} \frac{-4}{(2m+1)^2}\cdot\cos(2m+1)x$.

As $f(x) = \begin{cases} -x & \text{on } [-\pi,0] \\ x & \text{on } [0,\pi] \end{cases}$ is 2π-periodic and continuous, we can compare the values at $x=0$ of the Fourier series and $f(x)$. $\Rightarrow \frac{\pi}{2} - \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{4}{(2m+1)^2} = 0 \quad \therefore \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$

5. Idea: $\|\cdot\|_2 = \langle\cdot,\cdot\rangle$ and we know that $\cos nx$, $\sin nx$ are all orthogonal to each other.

Let $T_n(x) = A + \sum_{m=1}^{n} a_m\cos mx + \sum_{m=1}^{n} b_m\sin mx$. Then, $\|T_n\|_2 = \sqrt{\langle T_n, T_n\rangle} = \left(\langle A,A\rangle + \sum_{m=1}^{n}\langle a_m\cos mx, a_m\cos mx\rangle + \sum_{m=1}^{n}\langle b_m\sin mx, b_m\sin mx\rangle\right)^{1/2}$ (This is Pythagorean Theorem). However, $\langle\cos mx,\cos mx\rangle = \int_{-\pi}^{\pi}\cos^2 mx\,dx = \int_{-\pi}^{\pi}\frac{1-\cos 2mx}{2}dx = \pi - \frac{\sin 2mx}{4m}\Big|_{-\pi}^{\pi} = \pi$. Similarly, $\langle\sin mx,\sin mx\rangle = \pi$. But, $\langle 1,1\rangle = \int_{-\pi}^{\pi}1^2 dx = 2\pi$.

$\therefore \|T_n\|_2 = \sqrt{2\pi A^2 + \pi\cdot\sum_{m=1}^{n}(a_m^2+b_m^2)} = \sqrt{\pi}\cdot\sqrt{\frac{1}{2}(2A)^2 + \sum_{m=1}^{n}(a_m^2+b_m^2)}$.

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6. Idea: Apply the theorems you learned appropriately.

We are going to use theorems from the lecture note.

1) Weierstraß M-test tells us that $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx =: g(x)$ converges uniformly and absolutely on $[-\pi,\pi]$ b/c $|a_n \cos nx| < 2 \cdot |a_n|$, $|b_n \sin nx| < 2 \cdot |b_n|$ and $\sum |a_n| + \sum |b_n| < \infty$.

2) For the function $g(x)$ we obtained in 1), by applying Theorem 7, we get $f(x) = g(x)$ for $x \in [-\pi,\pi]$. So, they are exactly the same function. Going back to 1), we can conclude that the Fourier series of f converges absolutely and uniformly to g which is f.

7. Idea: Define a sequence of trigonometric polynomials T_n by considering $\varepsilon = \frac{1}{n}$.

(a) For $\varepsilon = \frac{1}{n} > 0$, say T_n is a trigonometric polynomial such that $|f(x) - T_n(x)| < \varepsilon$ for all $x \in \mathbb{R}$. So, $\|T_n - f\|_\infty \le \frac{1}{n}$. This implies that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ (for example, N can be a natural number larger than $\frac{1}{\varepsilon}$) s.t. $\forall n \ge N$, $\|T_n - f\|_\infty < 2\varepsilon$. Therefore, by Problem 1, $T_n \to f$ uniformly.

(b) We want to prove that $f(x) = f(x+2\pi)$ for all $x \in \mathbb{R}$.

We know that $T_n(x)$ converges to $f(x)$ and T_n is 2π-periodic. Therefore, $f(x) = \lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} T_n(x+2\pi) = f(x+2\pi)$. ($\cos nx, \sin nx$ are 2π-periodic and so is their sum)

Continuity follows from the fact that uniform convergence preserves continuity & sin, cos's are conti.

8. Idea: Apply the method from the book.

Refer to Chapter 6.2 Example 1 of Shearer & Levy. If we let $u(x,t) = v(x) \cdot w(t)$, then there is an eigenvalue λ s.t. $v'' + \lambda v = 0$ and $w' + 4\lambda w = 0$. Under the boundary condition, we get $v(0) = v(\pi) = 0$. Therefore, $v_n(x) = \sin nx$ and $w_n(t) = e^{-4n^2 t}$.

Now, we can consider $\sum_{n=1}^{\infty} c_n \sin nx \cdot e^{-4n^2 t}$ as a candidate. This satisfies the first two rows.

For the third part (= Initial condition), we can plug in $t=0 \Rightarrow c_1 = 1$ and $c_5 = -3$. $c_m = 0$ o.w.

$\therefore u(x,t) = e^{-4t} \sin x - 3e^{-100t} \cdot \sin 5x$.

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9. Idea: Same as Problem 8 except that it is the wave equation not the heat equation.

Refer to Claim 6.3 in Chapter 6.2 of Shearer & Levy.

We know that the solution is $\sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}$ using Separation of Variables.

Here, $a_n = \frac{2}{L}\int_0^L u(x,0)\cdot\sin\frac{n\pi x}{L}\,dx$ and $b_n = \frac{2}{n\pi c}\int_0^L u_t(x,0)\cdot\sin\frac{n\pi x}{L}\,dx$. In this problem, $c=3$, $L=1$, $u(x,0) = 2\sin\pi x + 7\sin 3\pi x$, and $u_t(x,0) = 2\sin\pi x$.

So, $a_n = 2\int_0^1 (2\sin\pi x + 7\sin 3\pi x)\cdot\sin n\pi x\,dx$. However, $\int_0^1 \sin m\pi x\sin n\pi x\,dx = \int_0^1 \frac{1}{2}[\cos((m-n)\pi x) - \cos((m+n)\pi x)]\,dx = 0$ if $m\neq n$ because $\int_0^1 \cos k\pi x\,dx = 0$ for any $k\neq 0, k\in\mathbb{Z}$. If $m=n$, then we get $\frac{1}{2}$. $\therefore a_1 = 2$, $a_3 = 7$, and $a_n = 0$ o.w.

Similarly, $b_n = \frac{2}{3n\pi}\int_0^1 2\sin\pi x\cdot\sin n\pi x\,dx = \frac{2}{3\pi}$ if $n=1$ and 0 o.w.

Therefore, $u(x,t) = \left(2\cos 3\pi t + \frac{2}{3\pi}\sin 3\pi t\right)\cdot\sin\pi x + 7\cdot\cos 9\pi t\cdot\sin 3\pi x$.