Homework 5 - Spring 2020 MATH 126-001 - Introduction to PDEs

1. Derive the 2D heat equation

$$u_t = k(u_{xx} + u_{yy}) = k\Delta u$$

over a domain $\Omega \in \mathbf{R}^2$ by invoking the Divergence Theorem. Let $\mathbf{f}$ be the heat flux over $\partial\Omega$ and assume Fourier's Law, that heat flows in the direction of steepest descent.

2. Let

$$h(x) = xe^{-x^2}.$$

Use the definition of the convolution operator to find

$$(h * h)(x).$$

3. Let $f, g \in L^1$ prove

$$\text{supp}(f * g) \subset \overline{\text{supp}(f) + \text{supp}(g)}$$

where

$$\text{supp}(f) + \text{supp}(g) = \{x + y \mid x \in \text{supp}(f), y \in \text{supp}(g)\}.$$

4. Write the solution of the Cauchy problem for the heat equation

$$u_t = ku_{xx}, \quad -\infty < x < \infty,\ t > 0,$$

with initial condition $u(x, 0) = \frac{1}{2}(H(x+1) - H(1-x))$ in terms of the error function

$$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy.$$

5. Solve the heat equation where

$$u(x,0) = \begin{cases} 0, & x < 0 \\ 10 - x, & x \geq 0 \end{cases}$$

and express your solution in terms of the error function from problem (4).

6. Consider the Cauchy problem from (4), with initial condition $u(x, 0) = x^2$.

   (a) Show that if $u(x, t)$ is the solution, then $v(x, t) = u_{xxx}(x, t)$ satisfies the heat equation with $v(x, 0) = 0$.

   (b) Find $u(x, t)$ as an explicit formula.

7. Formulate and prove a statement regarding conservation of energy for the wave equation on a bounded domain in $\mathbb{R}^n$:

$$\begin{aligned} u_{tt} &= c^2 \Delta u, \quad \mathbf{x} \in U,\ t > 0, \\ u(\mathbf{x}, 0) &= \phi(\mathbf{x}), \quad \mathbf{x} \in U, \\ u_t(\mathbf{x}, 0) &= \psi(\mathbf{x}), \quad \mathbf{x} \in U, \end{aligned}$$

under homogeneous Dirichlet or Neumann boundary conditions.

8. Use the maximum principle for the heat equation to prove theorem 5.2 from the textbook.

9. Let $u(x, 0) \leq v(x, 0)$ for all $x \in [a, b]$ and assume u and v are solutions to the homogeneous heat equation. For $t_0 > 0$, show

$$u(x, t_0) \leq v(x, t_0)$$

for all $x \in [a, b]$.

Homework 5 Solution

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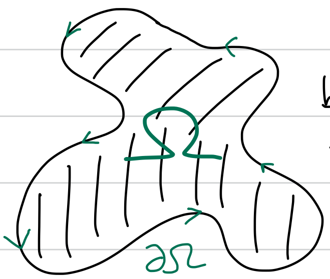
1. Idea: View the change of the heat energy with respect to the time t in two different perspectives.



Fourier's law (the heat flows in the direction of the steepest descent) can be expressed as "the heat flux at the boundary $\partial\Omega$ is $\underset{\text{constant}}{-K_0} \cdot \underset{\text{gradient of } u}{\nabla u} \underset{\text{dot product}}{\cdot} \nu$" ($\nu$: outward normal direction)

This implies that the heat disappearing from Ω should be the sum(= integral) of $-K_0 \cdot \nabla u \cdot \nu$.

On the other hand, we can simply take the derivative by t of the heat energy in Ω. The energy is (using heat dynamics) $c \cdot \rho \cdot u$ where c is the specific heat and ρ is the density. Therefore, the change of the energy with respect to the time will be $\frac{d}{dt}\iint_\Omega c\rho u \, dS$. Now, we get

By putting this $(-)$ sign, we are measuring the amount of heat loss.

$$-\underbrace{\frac{d}{dt}\iint_\Omega c\rho u \, dS}_{} = \underbrace{-K_0 \int_{\partial\Omega} \nabla u \cdot \nu \, ds}_{}$$

(Note that $-K_0 \nabla u \cdot \nu$ is the amount escaping, so the signs match)

$$-\iint_\Omega \frac{\partial}{\partial t}(c\rho u)\, dS \quad = \quad = -K_0 \int_{\partial\Omega} (u_x, u_y)\cdot(dy, -dx)$$

$$-\iint_\Omega c\rho u_t \, dS \quad = \quad = -K_0 \int_{\partial\Omega} (u_x dy - u_y dx)$$

$$= -K_0 \iint_\Omega (u_{xx} + u_{yy})\, dS$$

$$\therefore \iint_\Omega (c\rho u_t - K_0 \Delta u)\, dS = 0. \qquad = \iint_\Omega -K_0 \cdot \Delta u \, dS$$

However, this should be true for any small region $\Rightarrow c\rho u_t = K_0 \Delta u \Rightarrow u_t = k\Delta u$ for $k = \frac{K_0}{c \cdot \rho}$.

2. You can just do it. (The purple color integral computation has been corrected from the original version.)

$f * g\,(x) = \int_{-\infty}^{\infty} f(x-y) g(y)\, dy$. Plugging in $f = g = h$, we get

$$h * h\,(x) = \int_{-\infty}^{\infty} (x-y)\cdot e^{-(x-y)^2} \cdot y \cdot e^{-y^2}\, dy$$

$y = \frac{x}{2} + z$ OR $z = -\frac{x}{2} + y \Rightarrow dz = dy$

and $-\infty < y < \infty \Rightarrow -\infty < z < \infty$.

$$= \int_{-\infty}^{\infty} \left(x - \left(z + \tfrac{x}{2}\right)\right) e^{-\left(x - \left(z + \frac{x}{2}\right)\right)^2} \left(z + \tfrac{x}{2}\right) e^{-\left(z + \frac{x}{2}\right)^2} dz$$

$$= \int_{-\infty}^{\infty} \left(\frac{x}{2} - z\right) \cdot e^{-\left(\frac{x}{2} - z\right)^2} \left(\frac{x}{2} + z\right) e^{-\left(\frac{x}{2} + z\right)^2} dz$$

$$= \int_{-\infty}^{\infty} \left(\frac{x^2}{4} e^{-\left(\frac{x^2}{2} + 2z^2\right)} - z^2 \cdot e^{-\left(\frac{x^2}{2} + 2z^2\right)}\right) dz$$

$$= \frac{x^2}{4} \cdot e^{-\frac{x^2}{2}} \cdot \underbrace{\int_{-\infty}^{\infty} e^{-2z^2} dz}_{\underbrace{\int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{\sqrt{2}}}_{\sqrt{\frac{\pi}{2}}}} - e^{-\frac{x^2}{2}} \underbrace{\int_{-\infty}^{\infty} z^2 \cdot e^{-2z^2} dz}_{\frac{\sqrt{\pi}}{4\sqrt{2}}}.$$

You can use $\int u v' = uv - \int u' v$ carefully for $\int_0^\infty z \cdot z e^{-2z^2} dz$.

OR $\int_{-\infty}^{\infty} z^2 \cdot e^{-2z^2} dz$

$$= \int_{-\infty}^{\infty} -\frac{\partial}{\partial t}\left(e^{-tz^2}\right)\Big|_{t=2} dz$$

$$= \frac{d}{dt}\Big(\underbrace{-\int_{-\infty}^{\infty} e^{-tz^2} dz}_{-\sqrt{\frac{\pi}{t}}}\Big)\Big|_{t=2} = +\frac{1}{2}\sqrt{\frac{\pi}{t^3}}\Big|_{t=2}$$

$$= \frac{\sqrt{\pi}}{4\sqrt{2}}.$$

$$= \frac{\sqrt{\pi}}{4\sqrt{2}} x^2 e^{-\frac{x^2}{2}} - \frac{\sqrt{\pi}}{4\sqrt{2}} \cdot e^{-\frac{x^2}{2}}$$

$$= \frac{\sqrt{\pi}}{4\sqrt{2}} (x^2 - 1) \cdot e^{-\frac{x^2}{2}}.$$

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3. Idea: $x_0 \notin supp(f)$ if and only if $\exists \varepsilon > 0$ st f vanishes on $B(x_0, \varepsilon) := \{x \text{ s.t. } |x - x_0| < \varepsilon\}$.

Moreover, $supp(f)$ is closed and $\overline{supp(f) + supp(g)}$ is also closed.

Suppose $x_0 \notin \overline{supp(f) + supp(g)}$. As $\overline{supp(f) + supp(g)}$ is closed, $\exists \varepsilon > 0$ st. $B(x_0, \varepsilon) \cap (\overline{supp(f) + supp(g)}) = \phi$.

Now, $f * g\,(x) = \int_{-\infty}^{\infty} f(x-y) g(y)\, dy$. Now, I claim that if $g(y) \neq 0$, then $f(x-y) = 0$ for $x \in B(x_0, \varepsilon)$.

Why? $g(y) \neq 0$ implies that $y \in supp(g)$. But if $f(x-y) \neq 0$, then $x - y \in supp(f)$. Then,

$x = x - y + y \in supp(f) + supp(g)$, but $x \in B(x_0, \varepsilon)$ and $B(x_0, \varepsilon) \cap (\overline{supp(f) + supp(g)}) = \phi$. This is contradiction.

So, we have proven that $g(y) = 0$ OR (if $g(y) \neq 0$, then) $f(x-y) = 0$ for all $x \in B(x_0, \varepsilon)$.

Hence, $f(x-y) \cdot g(y) = 0$ for all $y \in \mathbb{R}$ so that $f * g\,(x) = 0$. Therefore, $f * g$ vanishes on $B(x_0, \varepsilon)$.

Applying the fact mentioned above, we get $x_0 \notin supp(f * g)$ $\therefore$ $supp(f * g) \subseteq \overline{supp(f) + supp(g)}$.

↯ There is a shorter proof: if $f * g(x_0) \neq 0$, then $\int_{\mathbb{R}} f(x_0 - y) g(y)\, dy \neq 0$. So, for some $y_0 \in \mathbb{R}$, $f(x_0 - y_0) \cdot g(y_0) \neq 0$ so that $x_0 - y_0 \in supp\, f$ and $y_0 \in supp\, g$ and, as a result, $x_0 = x_0 - y_0 + y_0$ belongs to $supp f + supp\, g$. This proves that $\{x \in \mathbb{R} : f * g(x) \neq 0\} \subseteq supp f + supp\, g$. Now, you can take the closure to get the result! [Thanks to Weizheng Yue for this proof.]

4. Idea: Apply the solution $\Phi(\ ,t) * g$ where g is the initial condition.

We know that $u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y, t)\, u(y, 0)\, dy$ is the unique solution.

$$= \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot \frac{1}{2}\big(H(y+1) - H(1-y)\big)\, dy.$$

Note that $H(y+1) - H(1-y) = 0$ for $-1 < y < 1$ & $\begin{cases} +1 & \text{if } y > 1 \\ -1 & \text{if } y < -1. \end{cases}$

$$= \frac{1}{\sqrt{4k\pi t}} \left[\int_{-\infty}^{-1} e^{-\frac{(x-y)^2}{4kt}} \cdot \left(-\frac{1}{2}\right) dy + \int_{1}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot \frac{1}{2}\, dy. \right]$$

We will substitute y by $-z + 2x$ in the first integration:

Then, $\int_{-\infty}^{-1} e^{-\frac{(x-y)^2}{4kt}} \cdot \left(-\frac{1}{2}\right) dy = \int_{\infty}^{2x+1} e^{-\frac{(z-x)^2}{4kt}} \cdot \left(-\frac{1}{2}\right) \cdot -dz = \int_{2x+1}^{\infty} e^{-\frac{(z-x)^2}{4kt}} \cdot \left(-\frac{1}{2}\right) dz.$

Hence, from $2x+1$ to ∞, the two integrations cancel out.

$$\therefore\ u(x,t) = \frac{1}{\sqrt{4k\pi t}} \cdot \frac{1}{2} \int_{1}^{2x+1} e^{-\frac{(y-x)^2}{4kt}} dy = \text{"} \int_{-x+1}^{x+1} e^{-\frac{y^2}{4kt}} dy$$

$$= \frac{1}{4\sqrt{k\pi t}} \cdot \int_{\frac{-x+1}{\sqrt{4kt}}}^{\frac{x+1}{\sqrt{4kt}}} e^{-z^2} d\left(\sqrt{4kt}\, z\right) = \frac{1}{2} \cdot \frac{1}{\sqrt{\pi}} \left[\int_0^{\frac{x+1}{\sqrt{4kt}}} e^{-z^2} dz - \int_0^{\frac{-x+1}{\sqrt{4kt}}} e^{-z^2} dz \right]$$

$$= \frac{1}{4}\left(Erf\left(\frac{x+1}{\sqrt{4kt}}\right) - Erf\left(\frac{-x+1}{\sqrt{4kt}}\right)\right).$$

↯ $H(x)$ is the heavyside function defined in Ch3 Problem 12 of Shearer & Levy.

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5. Just apply $u(x,t)=\int_{-\infty}^{\infty}\Phi(x-y,t)\,g(y)dy$.

$$u(x,t)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{4k\pi t}}e^{-\frac{(x-y)^2}{4kt}}u(y,0)dy$$
$$=\int_{0}^{\infty}\ \ "\ \ \ "\ \ (10-y)dy \quad (\text{b/c } u(y,0)=0 \text{ if } y<0.)$$

We substitute $x-y$ by z to get something looking like $\int e^{-z^2}dz$.

$$=\int_{x}^{-\infty}\frac{1}{\sqrt{4k\pi t}}\cdot e^{-\frac{z^2}{4kt}}(z+10-x)\,d(-z).$$
$$=\int_{-\infty}^{x}\frac{1}{\sqrt{4k\pi t}}\cdot e^{-\frac{z^2}{4kt}}(z+10-x)\,dz.$$

Here, $\int e^{-\frac{z^2}{4kt}}\cdot z=-2kt\cdot e^{-\frac{z^2}{4kt}}$, so the term with z gives $\sqrt{\frac{kt}{\pi}}\cdot e^{-\frac{x^2}{4kt}}$.

The remaining term is simply $\frac{10-x}{\sqrt{4k\pi t}}\cdot\int_{-\infty}^{x}e^{-\frac{z^2}{4kt}}dz$ and if we substitute $z=\sqrt{4kt}\cdot s$

the integration part becomes $\sqrt{4kt}\cdot\int_{-\infty}^{\frac{x}{\sqrt{4kt}}}e^{-s^2}ds=\sqrt{4kt}\left(\underbrace{\frac{\sqrt{\pi}}{2}Erf\left(\frac{x}{\sqrt{4kt}}\right)}_{\text{from } 0 \text{ to } \frac{x}{\sqrt{4kt}}}+\underbrace{\frac{\sqrt{\pi}}{2}}_{\text{from } -\infty \text{ to } 0}\right)$.

• Note that $\frac{\sqrt{\pi}}{2}Erf(sth)=\int_{0}^{sth}e^{-y^2}dy$

Combining all of them, we get

$$u(x,t)=\sqrt{\frac{kt}{\pi}}\,e^{-\frac{x^2}{4kt}}+\frac{10-x}{2}Erf\left(\frac{x}{\sqrt{4kt}}\right)+\frac{10-x}{2}.$$

6. Idea: Clairaut's theorem for part a. Part b ⇒ "$f_{xx}=0$ means $f(x,t)=x\cdot g(t)+h(t)$."

(a) u satisfies $u_t=ku_{xx}$.

For $v:=u_{xxx}$, $v_t=u_{xxxt}=u_{txxx}=ku_{xxxxx}=kv_{xx}$.

Clairaut's theorem (assuming u is C^4)

$u(x,0)=x^2$ implies $v(x,0)=u_{xxx}(x,0)=0$.

(b) As $v(x,0)=0$, $v(x,t)=\int_{-\infty}^{\infty}\Phi(x-y,t)\,v(y,0)dy=0$.

Hence, $u_{xxx}(x,t)=0 \Rightarrow u(x,t)=g_2(t)\,x^2+g_1(t)x+g_0(t)$.

$u_t=ku_{xx}$ implies $g_2'(t)x^2+g_1'(t)x+g_0'(t)=k\cdot g_2(t)\cdot 2$.

So, $g_2'(t)=0$, $g_1'(t)=0$, $g_0'(t)=2k\cdot g_2(t)$.

$\Downarrow$ $g_2(t)$ is constant, $\Downarrow$ $g_1(t)$ is constant.

Initial condition tells you

& $\underbrace{g_2(0)}_{=1}x^2+\underbrace{g_1(0)}_{=0}x+\underbrace{g_0(0)}_{=0}=x^2$.

Combining these information, we get $g_2(t)=1$, $g_1(t)=0$, $g_0(t)=2kt$.

Therefore, $u(x,t)=x^2+2kt$.

↯ Technical comment on part a: u should be C^4 to apply Clairaut's theorem and it should be five times differentiable since we need u_{xxxxx}.

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7. Idea: Mimic the formula for the energy function $E(t)$ and prove that $\frac{d}{dt}E(t)=0$.

Statement: Let u be a solution for the wave equation

$$u_{tt}=c^2\cdot\Delta u \quad w/ \quad u(x,0)=\phi(x) \quad \text{and} \quad u_t(x,0)=\psi(x) \quad \text{for } x\in U \text{ and } t>0.$$

Moreover, assume that u satisfies the homogeneous Dirichlet or Neumann boundary condition.

Let's define the energy function $E(t)$ as follows:

$$E(t)=\int_U \left(\frac{1}{2}u_t^2+\frac{c^2}{2}\left(\sum_{i=1}^{n}u_i^2\right)\right)dV$$

where u_i denotes $\frac{\partial u}{\partial x_i}$. Then, $E(t)$ is constant.

Proof. Let's take the derivative of $E(t)$ by t.

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_U \frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2+\frac{c^2}{2}\left(\sum_{i=1}^{n}u_i^2\right)\right)dV_n\\
&=\int_U\left(u_t\cdot u_{tt}+c^2\cdot\sum_{i=1}^{n}u_i\,u_{it}\right)dV_n \quad \left(u_{it} \text{ is } \frac{\partial^2 u}{\partial t\,\partial x_i}.\right)\\
&=\int_U\left(u_t\cdot c^2\cdot\Delta u+c^2\sum_{i=1}^{n}u_i u_{it}\right)dV_n \quad \left(\Delta u \text{ is } \sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}.\right)\\
&=c^2\cdot\int_U\sum_{i=1}^{n}\left(u_{ii}\cdot u_t+u_i\cdot u_{it}\right)dV_n \quad \left(u_{ii} \text{ is } \frac{\partial^2 u}{\partial x_i^2}.\right)\\
&=c^2\cdot\int_U\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(u_i\cdot u_t\right)dV_n\end{aligned}$$

Here, we will apply the divergence theorem to the domain U and the vector field defined by $\vec{F}=(u_1\cdot u_t,\cdots,u_n\cdot u_t)$ $\left(\text{OR } = u_t\cdot(u_1,\cdots,u_n)=u_t\nabla u\right)$

$$\underbrace{\int_U \operatorname{div}\vec{F}\,dV_n}_{\int_U\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(u_i\cdot u_t)\,dV_n}=\underbrace{\int_{\partial U}\vec{F}\cdot\vec{n}\,dV_{n-1}}_{=\int_{\partial U}u_t\cdot\nabla u\cdot\vec{n}\,dV_{n-1}} \quad \text{dot product.}$$

Now, it is enough to prove that the right hand side is zero under homogeneous Dirichlet / Neumann BC.

i) Homogeneous Dirichlet condition : $u(x,t)=0$ on ∂U. So, $u_t(x,t)=0$ on $x\in\partial U$.

$$\text{Hence, } \int_{\partial U}\underset{0}{\underset{\parallel}{u_t}}\cdot\nabla u\cdot\vec{n}\,dV_{n-1}=0.$$

ii) Homogeneous Neumann condition : $\frac{\partial u}{\partial \nu}=0$ on ∂U. However, $\frac{\partial}{\partial \nu}$ is the $\vec{n}$-directional derivative, that is, $\frac{\partial u}{\partial \nu}=(u_1,\cdots,u_n)\cdot\vec{n}=\nabla u\cdot\vec{n}$.

$$\text{So, } \int_{\partial U}u_t\cdot\underbrace{\nabla u\cdot\vec{n}}_{=0}\,dV_{n-1}=0.$$

In both cases, we get 0 on the right hand side. Therefore,

$$\frac{d}{dt}E(t)=c^2\cdot\int_U\operatorname{div}\vec{F}\,dV=c^2\cdot 0=0.$$

↳ Dirichlet boundary condition OR Neumann boundary condition can be found in Section 1.2 of Shearer & Levy. (For Neumann condition, $\frac{\partial u}{\partial \nu}(x)$ means $\lim_{h\to 0}\frac{u(x+h\vec{\nu})-u(x)}{|h\vec{\nu}|}$.)

DongGyu Lim

8. Idea : Think about the difference of two solutions.

Assume that we have two solutions u_1 and u_2. Our goal is to prove that $u_1 = u_2$.
In other words, it is enough to prove that $u_1 - u_2 \equiv 0$.
Let $v(x,t) := u_1(x,t) - u_2(x,t)$. Then, $v(x,t)$ satisfies the following equations :

$v_t = k\Delta v$, $\quad v(x,0) = u_1(x,0) - u_2(x,0) = \phi(x) - \phi(x) \equiv 0$, $\quad v_t(x,0) = u_{1,t}(x,0) - u_{2,t}(x,0) = \psi(x) - \psi(x) \equiv 0$,

and, in Dirichlet case, $v(x,t) = u_1(x,t) - u_2(x,t) = f(x) - f(x) \equiv 0$ on $x \in \partial U$, $t > 0$.
Hence, it is enough to prove that homogeneous heat equations w/ initial condition zero functions and homogeneous boundary conditions should have the zero function as the unique solution.

Dirichlet : $v \equiv 0$ on the boundary. However, according to the maximum principle, this implies that $v(x,t) \le 0$ for all $x \in U$, $t > 0$. On the other hand, $-v$ satisfies exactly the same equation $\Rightarrow -v(x,t) \le 0 \Rightarrow 0 \le v(x,t)$.
Therefore, $v(x,t) \equiv 0$, that is, $u_1(x,t) = u_2(x,t)$. So, the solution is unique.

↯ I have solved the n-dim'l version. For the 1-dim'l heat equation, you can even solve the Neumann boundary case.

9. Idea: Consider $v(x,t) - u(x,t)$ $(=: w(x,t))$.

$w(x,t)$ satisfies $w_t = k w_{xx}$ and $w(x,0) \ge 0$ for all $x \in [a,b]$. We need to prove that $w(x,t) \ge 0$ for any $x \in [a,b]$ and $t > 0$. Since they are solutions for the same heat equation, $w(a,t) = 0$ and $w(b,t) = 0$. We have the minimum principle as well as the maximum principle, that is, the minimum should be obtained on $\{a\} \times [0,T] \cup [a,b] \times \{0\} \cup \{b\} \times [0,T]$. [This is called the parabolic boundary.]

Over here, $w(x,t) = 0$ (boundary condition). Over here, $w(x,t) \ge 0$.

So, the minimum on the parabolic boundary is zero and, as a result, $w(x,t) \ge 0$ on $[a,b] \times [0,T]$.
We can choose T arbitrary, so $w(x,t) \ge 0$ for all $x \in [a,b]$ and $t > 0$.

↯ Note that if the interval is not $[a,b]$ but $(-\infty, \infty)$, we cannot solve this problem. Note that the 1-dim'l heat equation over the real line $(-\infty, \infty)$ w/ the initial condition $u(x,0) = 0$ does NOT have only unique solution.
You can search for Tychonoff's example : For $g(t) = \begin{cases} e^{-1/t^2} & t \ge 0 \\ 0 & < 0 \end{cases}$, $u(x,t) = \sum_{k=1}^{\infty} \frac{g^{(2k)}(t)\, x^{2k}}{(2k)!}$.