Homework 4 - Spring 2020 MATH 126-001 - Introduction to PDEs

1. Consider the wave equation that includes frictional damping:

$$u_{tt} + \mu u_t = c^2 u_{xx},$$

in which $\mu > 0$ is a damping constant. Show that if $u(x,t)$ is a C^2 solution with $u_x \to 0$ as $x \to \infty$, then the total energy $E(t) = \int_{-\infty}^{\infty} \frac{1}{2}(u_t^2 + c^2 u_x^2)dx$ is a decreasing function.

Incidentally, can you devise a C^2 function $f(x)$ with the property $f(x)$ approaches a constant as $x \to \pm\infty$, but $f'(x)$ does not approach zero?

2. Consider the second order ODE

$$\begin{aligned} u''(t) + c^2 u(t) &= f(t) \\ u(0) &= \phi \\ u_t(0) &= \psi. \end{aligned}$$

where $u : \mathbf{R} \to \mathbf{R}$ is C^2 and $\phi, \psi \in \mathbf{R}$.

(a) Express the second order ODE as a system of first order ODE

$$\begin{aligned} \mathbf{U}_t + A\mathbf{U} &= \mathbf{F} \\ \mathbf{U}(0) &= \Phi \end{aligned}$$

where $\mathbf{U}$, $\mathbf{U}_t$, and $\mathbf{F}$ are vector valued functions in $\mathbf{R}^2$, $A \in \mathbf{R}^{2\times 2}$, $\Phi \in \mathbf{R}^2$.

(b) Solve the system from part (2a) and show that

$$\mathbf{U} = e^{-At} \int_0^t e^{As}\mathbf{F}(s)\, ds + \Phi e^{-At}.$$

(c) Define the solution operator S by

$$S(t)\mathbf{W} = e^{-At}\mathbf{W}$$

and express the solution from (2b) in terms of S, Φ and $\mathbf{F}$.
What ODE does $S(t)\Phi$ solve?

3. Let

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f(x,t), \quad x \in \mathbf{R}, t > 0 \\ u(x,0) &= \phi(x), \quad x \in \mathbf{R} \\ u_t(x,0) &= \psi(x), \quad x \in \mathbf{R}. \end{aligned}$$

(a) Express the second order PDE in terms of a system of PDEs as you did in (2a).

(b) Solve the system from (3a) for the case

$$\begin{aligned} u_{tt} - c^2 u_{xx} = 0, \quad & x \in \mathbf{R}, t > 0 \\ u(x,0) = \phi(x), \quad & x \in \mathbf{R} \\ u_t(x,0) = \psi(x), \quad & x \in \mathbf{R}. \end{aligned}$$

and define the solution as $\mathbf{U}_h$. The solution operator for the system is therefore $S(t)\Phi(x) = \mathbf{U}_h$.

(c) Following the pattern established in (2c), write the solution of the non homogeneous system established (3a) and compare it to the solution we derived in Section 4.4.

4. Find the solution of

$$\begin{aligned} u_{tt} - u_{xx} = f(x,t), \quad & x > 0, t > 0 \\ u(x,0) = \phi(x), \quad & x > 0 \\ u_t(x,0) = \psi(x), \quad & x > 0 \\ u(0,t) = \phi(0) = 0 & \end{aligned}$$

for $x \leq t$ by using Green's Theorem and integrating over the domain of dependence.

5. Consider the wave equation in three dimensions, with initial conditions in which $\phi(\mathbf{x}) = f(|\mathbf{x}|)$ is rotationally symmetric, the function f satisfies $f(r) = 0$, $r \geq \epsilon$, and $\psi \equiv 0$. Show that the solution $u(\mathbf{x}, t)$ is (a) rotationally symmetric, and (b) zero outside a circular strip centered at the origin and having width ϵ.

6. Show that $\Delta\phi(r,t) = \phi_{rr} + \frac{n-1}{r}\phi_r$. Consequently, the heat equation for rotationally symmetric functions $u(\mathbf{x},t) = \phi(r,t)$, $r = |\mathbf{x}|$, is

$$\phi_t = k\left(\phi_{rr} + \frac{n-1}{r}\phi_r\right).$$

Also do the same problem but for the wave equation in $\mathbf{R}^3$, show $u(r,t)$ satisfies

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right).$$

7. (a) Let $g : [0,\infty) \to \mathbb{R}$ be a bounded integrable function. Prove directly that

$$u(x,t) = \int_0^\infty (\Phi(x-y,t) - \Phi(x+y,t))g(y)dy$$

is an odd function of $x \in \mathbb{R}$ for each $t > 0$. (Here, $\Phi(x,t) := \frac{1}{\sqrt{4\pi kt}}e^{\frac{-x^2}{4kt}}$.)

(b) Let $h : \mathbb{R} \to \mathbb{R}$ be an odd bounded integrable function. Prove that

$$u(x,t) = \int_{-\infty}^\infty \Phi(x-y,t)h(y)dy$$

is an odd function of $x \in \mathbb{R}$ for each $t > 0$. That is, the symmetry in the initial data is carried through to the same symmetry in the solution.

Homework 4 Solution

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1. Idea: You can basically mimic the argument in 4.3.

It is enough to show that $E'(t)$ is negative.

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2}(u_t^2 + c^2u_x^2)\,dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{2}\frac{\partial}{\partial t}\left(u_t^2 + c^2 u_x^2\right)dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{2}\left(2u_t u_{tt} + 2c^2 u_x u_{xt}\right)dx$$
$$= \int_{-\infty}^{\infty} \left(u_t\cdot(c^2 u_{xx} - \mu u_t) + c^2 u_x u_{xt}\right)dx$$
$$= \int_{-\infty}^{\infty} -\mu\cdot u_t^2\,dx + c^2\int_{-\infty}^{\infty}\left(u_t\cdot u_{xx} + u_x u_{xt}\right)dx$$
$$= -\mu\cdot\int_{-\infty}^{\infty} u_t^2\,dx + c^2\int_{-\infty}^{\infty}\frac{\partial}{\partial x}(u_t\cdot u_x)\,dx$$
$$= -\mu\cdot(\text{positive number}) + 0 < 0.$$

What if $u_t \equiv 0$?
$\Rightarrow$ the equation becomes $u_{tt} = c^2 u_{xx}$.
$\Rightarrow u(x,t) = F(x+ct) + G(x-ct)$
$\Rightarrow F'(x+ct) = G'(x-ct)$ ← compute $u_t \equiv 0$.
$\Rightarrow F' = G' \equiv$ constant $\Rightarrow u(x,t) = C_0 x + C_1$.
But, $u_x \to 0$ as $x \to \infty \Rightarrow C_0 = 0$.
Therefore, $u(x,t)$ should be constant.

First of all, we can think about $f(x) = x\sin\frac{1}{x}$. We know that the derivative does not go to zero even though f goes to 0 (as x goes to 0). So, this famous function is giving some clue about what we are looking for. We want an example of 'as $x\to\infty$', so let's take $\frac{1}{x}\sin x$ instead. However, the derivative becomes $\frac{x\cos x - \sin x}{x^2}$ and it converges to 0 as $x\to\infty$. To make $x\cos x$ part have x^2-term, we can try $\frac{1}{x}\sin(x^2)$.
Now, $\left(\frac{1}{x}\sin(x^2)\right)' = \frac{x\cdot 2x\cdot\cos(x^2) - \sin(x^2)}{x^2} = 2\cos(x^2) - \frac{\sin(x^2)}{x^2}$ and now it does not converge as $x\to\infty$.

↯ It looks like you need some extra assumptions on u (or u_x & u_t). For example, (as in the argument of 4.3) we need to assume that u_t and u_x are L^2-functions. Otherwise, it is not necessarily the case that we can talk about $\int_{-\infty}^{\infty} u_t\cdot u_x$.

2. We learn this matrix method in Math 54.

(a) Let $U(t)$ be the vector function $\begin{pmatrix} u(t) \\ u'(t)\end{pmatrix}$. Then, $U_t = \begin{pmatrix} u'(t) \\ u''(t)\end{pmatrix} = \begin{pmatrix} u'(t) \\ -c^2u(t) + f(t)\end{pmatrix}$
$= \begin{pmatrix} 0 & 1 \\ -c^2 & 0\end{pmatrix}\begin{pmatrix} u(t) \\ u'(t)\end{pmatrix} + \begin{pmatrix} 0 \\ f(t)\end{pmatrix}$.
If we let A be $\begin{pmatrix} 0 & 1 \\ -c^2 & 0\end{pmatrix}$ and $F = \begin{pmatrix} 0 \\ f\end{pmatrix}$,
we get $U_t = AU + F$. Moreover, the initial condition becomes $U(0) = \begin{pmatrix} u(0) \\ u'(0)\end{pmatrix} = \begin{pmatrix} \phi \\ \psi\end{pmatrix} =: \Phi$.

(b) We introduce $e^{At} := I_2 + At + \frac{(At)^2}{2!} + \cdots$ which satisfies $(e^{At})' = A\cdot e^{At}$.
Then, $(e^{-At}\cdot U)' = e^{-At}U' - A\cdot e^{-At}U = e^{-At}(U_t - AU) = e^{-At}F$. (Here, $Ae^{-At} = e^{-At}A$ is used.)
Therefore, $e^{-At}\cdot U = \int_0^t e^{-As}\cdot F(s)\,ds + C$ and $C = \Phi$ can be obtained from the initial condition.

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(c) We can change all the terms with e^{At} into the ones with $S(t)$.

$$U = e^{-At}\int_0^t e^{As}F(s)ds + e^{-At}\cdot\Phi$$

$$= \int_0^t S(t-s)F(s)ds + S(t)(\Phi).$$

$S(t)(\Phi)$ solves the equation $U_t + AU = 0$ and $U(0) = \Phi$.

3. Idea: We regard u_{xx} to be $\partial_x^2(u)$ and consider ∂_x^2 as if it is a number.

(a) Let U be $\begin{pmatrix}u\\u_t\end{pmatrix}$. Then, $U_t = \begin{pmatrix}u_t\\u_{tt}\end{pmatrix} = \begin{pmatrix}u_t\\c^2u_{xx}+f\end{pmatrix} = \begin{pmatrix}u_t\\c^2\partial_x^2 u\end{pmatrix} + \begin{pmatrix}0\\f\end{pmatrix} = \begin{pmatrix}0&1\\c^2\partial_x^2&0\end{pmatrix}\begin{pmatrix}u\\u_t\end{pmatrix} + \begin{pmatrix}0\\f\end{pmatrix}$

$= -AU + F$.

The initial condition is $U(x,0) = \begin{pmatrix}\phi(x)\\\psi(x)\end{pmatrix}$.

(b) The unique solution of $u_{tt} = c^2u_{xx}$ w/ $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ $\forall x\in\mathbb{R}, t>0$ is given by d'Alembert solution: $u(x,t) = \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy$.

Hence, $S(t)\Phi(x)$ can be obtained by $\begin{pmatrix}u(x,t)\\u_t(x,t)\end{pmatrix} = \begin{pmatrix}\frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy\\ \frac{c}{2}[\phi'(x+ct)-\phi'(x-ct)] + \frac{1}{2}[\psi(x+ct)+\psi(x-ct)]\end{pmatrix}$.

(c) So, given any two functions of x: $\phi(x)$ and $\psi(x)$, we understand the operator $S(t)$ acting on $\Phi(x) = \begin{pmatrix}\phi(x)\\\psi(x)\end{pmatrix}$ as $S(t)(\Phi(x)) = \begin{pmatrix}\frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy\\ \frac{c}{2}[\phi'(x+ct)-\phi'(x-ct)] + \frac{1}{2}[\psi(x+ct)+\psi(x-ct)]\end{pmatrix}$ and then $S(t)(\Phi(x))$ satisfies the equation $U_t + AU = 0$.

Now, we want to solve the nonhomogeneous system and the strategy is the following:

Consider $U = S(t)(\Phi(x)) + \int_0^t S(t-s)(F(x,s))ds$.

Then, $U_t + AU = \underbrace{\left(S(t)(\Phi(x))_t + A\,S(t)(\Phi(x))\right)}_{=0 \text{ by definition of } S(t).} + \frac{\partial}{\partial t}\int_0^t S(t-s)(F(x,s))ds + A\cdot\int_0^t S(t-s)(F(x,s))ds$.

$$= S(t-t)F(x,t) + \underbrace{\int_0^t \left(S(t-s)(F(x,s))\right)_t ds + A\int_0^t S(t-s)(F(x,s))ds}_{\text{For any } s, \text{ they add up to zero.}}$$

$$= F(x,t) + 0.$$

So, $U(x,t) := S(t)(\Phi(x)) + \int_0^t S(t-s)(F(x,s))ds$ gives a solution to the nonhomogeneous equation. We know that it is unique. So, this should turn out to be the same as the one from 4.4.

Let's double check: Note that we only need to compute the first coordinate. $S(t)(\Phi(x))$ gives $\frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy$. Recall that $F(x,s) = \begin{pmatrix}0\\f(x,s)\end{pmatrix}$, so $S(t-s)(F(x,s))$ will have $\frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy$ as its first coordinate. Therefore, $u(x,t)$ is $\frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy\,ds$. They coincide.

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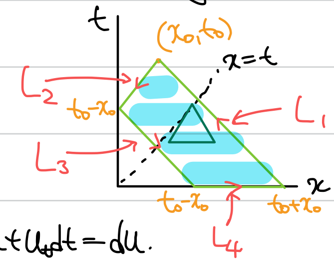
4. Idea: The point of 'using Green's theorem' is that we consider the region bounded by $x+ct=\text{const}.$ & $x-ct=\text{const}.$ and take the surface integral of $u_{tt}-c^2u_{xx}$.

We will apply Green's theorem to $\iint_\Delta f\,dxdt$ where Δ is the blue region below:

By Green's theorem $\iint_\Delta f\,dxdt = \iint_\Delta (u_{tt}-c^2u_{xx})\,dxdt$

$$= \int_{L_1\cup L_2\cup L_3\cup L_4} (-c^2u_x\,dt - u_t\,dx)$$



We separate the segments: (note that $c=1$ in our case)

L_1: $x+t=x_0+t_0$ ($\because$ constant) so $dx+dt=0$ and $-u_x\,dt-u_t\,dx = u_x\,dx+u_t\,dt = du$.

$$\int_{L_1} \text{ is just } \int du = u(x_0,t_0) - \underbrace{u(x_0+t_0,0)}_{=\phi(x_0+t_0)}.$$

L_2: $x-t = x_0-t_0<0$. Similarly, we get $\int_{L_2} = -\int du = u(x_0,t_0)-u(0,t_0-x_0)$.

L_3: $x+t=t_0-x_0$. Similarly, $\int_{L_3} = \int du = u(t_0-x_0,0) - u(0,t_0-x_0)$

$u(t_0-x_0,0) = \phi(t_0-x_0)$.

L_4: $t=0 \Rightarrow dt=0$, so $\int_{L_4} = \int -u_t\,dx = -\int_{t_0-x_0}^{t_0+x_0}\psi(x)\,dx$.

Therefore, $\iint_\Delta f\,dxdt = 2u(x_0,t_0) - \phi(x_0+t_0) + \phi(t_0-x_0) - 2\underbrace{u(0,t_0-x_0)}_{=0} - \int_{t_0-x_0}^{t_0+x_0}\psi(x)\,dx$.

$$\Rightarrow u(x_0,t_0) = \frac{1}{2}\left[\phi(x_0+t_0)-\phi(t_0-x_0)\right] + \frac{1}{2}\int_{t_0-x_0}^{t_0+x_0}\psi(y)\,dy + \frac{1}{2}\iint_\Delta f(x,t)\,dxdt.$$

Here, $\iint_\Delta f(x,t)\,dxdt = \int_0^{t_0}\int_{|t-(t_0-x_0)|}^{t_0+x_0-t} f(x,t)\,dxdt$.

$$u(x,t) = \frac{1}{2}\left[\phi(x+t)-\phi(t-x)\right] + \frac{1}{2}\int_{t-x}^{t+x}\psi(y)\,dy + \frac{1}{2}\int_0^t\int_{|(t-x)-s|}^{t+x-s} f(y,s)\,dyds.$$

↯ You need to be careful about the direction of the integrals.

5. Idea: $u(\mathbf{x},t)$ is obtained by integrating the initial condition function along a sphere of the center $\mathbf{x}$.

(a) From (4.23) in Section 4.5 of the textbook, we have the following formula for the solution:

$$u(\mathbf{x},t) = t\underbrace{\fint_{S(\mathbf{x},ct)}\psi(\mathbf{y})\,dS}_{=0 \text{ in our case}} + \frac{\partial}{\partial t}\left(t\cdot\fint_{S(\mathbf{x},ct)}\phi(\mathbf{y})\,dS\right)$$

It is enough to prove that $\fint_{S(\mathbf{x},ct)}\phi(\mathbf{y})\,dS$ only depends on $|\mathbf{x}|$ for a fixed t.

We will use a well-known fact from Linear Algebra. Let $\mathbf{x}'\in\mathbb{R}^3$ be another vector of the same length. Then, we can find a linear transformation T (or 3×3 matrix M)

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Satisfying the following conditions : 1) $M\mathbb{X} = \mathbb{X}'$ 2) $M^T M = I_3$ (the 3x3 identity matrix)
3) $\det M = 1$.

How? You can find an orthonormal basis $\mathcal{B}$ containing $\mathbb{X}$ and $\mathcal{B}'$ containing $\mathbb{X}'$ whose orientations are the same.
Then, you can consider the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{B}'$.

Clearly, M defines a differentiable bijective map from $S(\mathbb{X}, ct)$ to $S(\mathbb{X}', ct)$ because
$M(\mathbb{X} + ct\cdot\vec{v}) = M\mathbb{X} + ct\cdot M\vec{v} = \mathbb{X}' + ct\cdot\vec{v}'$ but M preserves the length and bijection
so $\vec{v}'$ covers $S(0,1)$.

Now, we apply the change of coordinates formula to our integration:

$$\int_{S(\mathbb{X}',ct)} \phi(\mathbb{Y}')\,dS' = \int_{v'\in S(0,ct)} \phi(\mathbb{X}' + v')\,dS'$$

dS' : the area given by the original parametrization.

translation does not change the area

change of coordinates $\mathbb{X}' = M\mathbb{X}$.

dS : the area after changing the coordinates

$$= \int_{v\in S(0,ct)} \phi(M\mathbb{X} + Mv)\underbrace{\frac{dS'}{dS}}_{=1}\,dS$$

because this term is computed by $\dfrac{|Mu \times Mv|}{|u\times v|}$ for some vectors u & v
but $Mu\times Mv = M(u\times v)$ because M preserves the length, the angle, and the orientation.

because $\phi(\mathbb{X}) = f(|\mathbb{X}|)$
$\Rightarrow \phi(M\mathbb{X}) = f(|M\mathbb{X}|)$
$= f(\sqrt{\mathbb{X}^T M^T M \mathbb{X}})$
$= f(\sqrt{\mathbb{X}^T\mathbb{X}})$
$= f(|\mathbb{X}|) = \phi(\mathbb{X})$.

$$= \int_{v\in S(0,ct)} \phi(\mathbb{X} + v)\,dS$$

$$= \int_{S(\mathbb{X},ct)} \phi(\mathbb{Y})\,dS$$

translation does not change dS.

Therefore, $u(\mathbb{X}, t)$ is constant as long as $\mathbb{X}$ varies over vectors of the same length.
So, it is rotationally symmetric.

(b) The circular strip means where $\mathbb{X}$ and t satisfies $B(\mathbb{X}, ct) \cap B(0,\varepsilon) = \phi$ where $B(\mathbb{X}, r)$ is the ball (not a sphere) centered at $\mathbb{X}$ with the radius r.

If we look at the integration formula, it is given by $\frac{\partial}{\partial t}\left(t\int_{S(\mathbb{X},ct)} \phi(\mathbb{Y})\,dS\right)$.

However, $\phi(\mathbb{Y}) = f(|\mathbb{Y}|)$ is zero for $\mathbb{Y} \notin B(0,\varepsilon)$ but $S(\mathbb{X},ct)\cap B(0,\varepsilon) \subseteq B(\mathbb{X},ct)\cap B(0,\varepsilon) = \phi$.

Therefore, the integration is zero. Hence, the derivative becomes also zero $\frac{\partial}{\partial t}(t\times 0) = 0$.

$\sharp$ (a) This looks quite simple. However, there seems to be no simple proof. Note that $S(\mathbb{X}, ct)$ is not of constant distance from the origin. So, you cannot just say $\int f(|\mathbb{Y}|)$ should all be equal. You really need a parametrization of two distinct spheres $S(\mathbb{X}, ct)$ and $S(\mathbb{X}', ct)$. (b) I think this problem is written in a very vague way because it does not specify if $|\mathbb{X}| > \varepsilon$ or $B(\mathbb{X}, ct) \subset B(0,\varepsilon)^c$. The meaning of the circular strip should be specified concretely. (This textbook seems to contain lots of bad problems :( )

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6. Idea: r is the function of $x_1, \cdots, x_n$ given by $r(x_1, \cdots, x_n) = \sqrt{x_1^2 + \cdots + x_n^2}$. So, $r_{x_i} = \frac{x_i}{r}$.

Δ is defined to be the sum of second partial derivatives. Let's first consider $\frac{\partial u}{\partial x_i}$.

$$\frac{\partial u}{\partial x_i} = \frac{\partial \phi}{\partial x_i} = \phi_r \cdot \frac{\partial r}{\partial x_i} + \phi_t \cdot \frac{\partial t}{\partial x_i} = \phi_r \cdot \frac{x_i}{r} + 0 = \frac{x_i}{r} \cdot \phi_r.$$

Now, $$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{x_i}{r} \cdot \phi_r\right) = \frac{r - x_i \cdot \frac{\partial r}{\partial x_i}}{r^2} \cdot \phi_r + \frac{x_i}{r} \cdot \left(\phi_{rr} \cdot \frac{\partial r}{\partial x_i} + \phi_{rt} \cdot \underbrace{\frac{\partial t}{\partial x_i}}_{=0}\right)$$
$$= \frac{r^2 - x_i^2}{r^3} \phi_r + \frac{x_i^2}{r^2} \phi_{rr}.$$

Taking the sum over $i = 1, \cdots, n$, we get

$$\Delta u (= \Delta \phi) = \frac{nr^2 - \sum_{i=1}^n x_i^2}{r^3} \phi_r + \frac{\sum_{i=1}^n x_i^2}{r^2} \phi_{rr}$$
$$= \frac{(n-1) r^2}{r^3} \phi_r + \phi_{rr} = \frac{n-1}{r} \phi_r + \phi_{rr}$$ (because $r = \sqrt{x_1^2 + \cdots + x_n^2}$)

u_t is just ϕ_t (Chain Rule). Therefore the heat equation for a rotationally symmetric function is

$$\phi_t = k\left(\phi_{rr} + \frac{n-1}{r} \phi_r\right).$$

For the wave equation, just remember that the right hand side of the equation is also the Laplacian of u, that is, Δu $(= u_{xx} + u_{yy} + u_{zz})$. We are doing it in $\mathbb{R}^3$, so $n=3$ and we get

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r} u_r\right).$$

7. You can just do it.

(a) Our goal is to prove $u(-x,t) = -u(x,t)$.

$$u(-x,t) = \int_0^\infty \left(\Phi(-x-y,t) - \Phi(-x+y,t)\right) g(y)\,dy = \int_0^\infty \left(\Phi(x+y,t) - \Phi(x-y,t)\right) g(y)\,dy$$
$$= -\int_0^\infty \left(\Phi(x-y,t) - \Phi(x+y,t)\right) g(y)\,dy = -u(x,t).$$

$\Phi(x,t) := \frac{1}{\sqrt{4\pi k t}} e^{\frac{-x^2}{4kt}}$, so
$\Phi(x,t) = \Phi(-x,t)$

(b) Similarly, let's consider $$u(-x,t) = \int_{-\infty}^\infty \Phi(-x-y,t) h(y)\,dy.$$
$$= \int_{-\infty}^\infty \Phi(x+y,t) h(y)\,dy.$$

We change coordinates $y \longleftrightarrow -y$: $$= \int_\infty^{-\infty} \Phi(x-y,t) h(-y)\,d(-y).$$

$-\int_{-\infty}^\infty = \int_\infty^{-\infty}$

$$= \int_\infty^{-\infty} \Phi(x-y,t) \cdot -h(y) \cdot -dy = -\int_{-\infty}^\infty \Phi(x-y,t) h(y)\,dy = u(-x,t).$$

Therefore, if the initial data is odd symmetric then so is the solution.