Homework 3 - Spring 2020 MATH 126-001 - Introduction to PDEs

1. Suppose in the traffic flow model discussed in section 2.4 that the speed v of cars is taken to be a positive monotonic differentiable function of density: $v = v(u)$.
   (a) Should v be increasing or decreasing?
   (b) How would you characterize the maximum velocity $v_{\max}$ and the maximum density $u_{\max}$?
   (c) Let $Q(u) = uv(u)$. Prove that Q has a maximum at some density u^* in the interval $(0, u_{\max})$.
   (d) Can there be two local maxima of the flux? (Hint: Make $Q(u)$ quartic.)

2. Carry through the analysis presented in section 3.4 for a general scalar conservation law
$$u_t + f(u)_x = 0$$
where $f : \mathbb{R} \to \mathbb{R}$ is a given C^2 function. Derive an implicit equation for the solution $u(x,t)$ of the Cauchy problem, and formulate a condition for the solution to remain smooth for all time. Likewise, if the condition is violated, find an expression for the time at which the solution first breaks down.

3. Suppose
$$u_t + (1 - 2u)u_x = 0$$
and
$$u(x,0) = \begin{cases} 1 & x < 0 \\ 1 - x & x \in (0,1) \\ 0 & x > 1 \end{cases}.$$
Find $u(x,t)$ and graph the characteristics for $x_0 < 0$, $x_0 \in (0,1)$ and $x_0 \geq 1$ in the xt-plane.

4. Suppose
$$u_t + uu_x = 0$$
and
$$u(x,0) = \begin{cases} 1 & x < 0 \\ 1 - x & x \in (0,1) \\ 0 & x > 1 \end{cases}.$$
Find $u(x,t)$ and graph the characteristics for $x_0 < 0$, $x_0 \in (0,1)$ and $x_0 \geq 1$ in the xt-plane.

5. Solve
$$u_{tt} = c^2 u_{xx}$$
where $u(x,0) = e^x$ and $u_t(x,0) = \sin(x)$ for all $x \in \mathbf{R}$.

6. Solve

$$u_{tt} = c^2 u_{xx}$$

where

$$u(x,0) = \begin{cases} 0 & x < -1 \\ 1 & x \in (-1,1) \\ 0 & x > 1 \end{cases}$$

and $u_t(x,0) = 0$ for all $x \in \mathbf{R}$.

7. Let ϕ and ψ be odd functions in x from the IVP for the wave equation. Is the solution $u(x,t)$ of the IVP for the wave equation even, odd or neither in the variable x for all t?

8. The midpoint of a piano string of tension T, density ρ, and length l is hit by a hammer whose head diameter is $2a$. A tiny fly is sitting at a distance $l/4$ from one end. (Assume that $a < l/4$; otherwise, the fly may be struck!) How long does it take for the disturbance to reach the fly?

9. Consider the initial value problem

$$\begin{aligned} u_{tt} &= u_{xx}, && -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= \phi(x), && -\infty < x < \infty, \\ u_t(x,0) &= \psi(x), && -\infty < x < \infty. \end{aligned}$$

Let $\phi(x)$ be the function defined by

$$\phi(x) = \begin{cases} 0 & x < 1 \\ x-1 & 1 \le x < 2 \\ 3-x & 2 \le x < 3 \\ 0 & 3 \le x \end{cases}$$

and $\psi(x) \equiv 0$. In the $x-t$ plane representation of the solution in Figure 4.5, we find that $u \equiv 0$ in the middle section, with $t > \frac{1}{2}$. Show that if we keep the same ϕ but make ψ nonzero, with $\text{supp}\phi = [1,3]$, then u will still be constant in the middle section. Find a condition on ψ that is necessary and sufficient to make this constant 0.

10. Consider C^3 solutions of the wave equation

$$u_{tt} = c^2 u_{xx}.$$

For $c = 1$, define the energy density $e = \frac{1}{2}(u_t^2 + u_x^2)$, and let $p = u_t u_x$ (the momentum density).

(a) Show that $e_t = p_x$ and $e_x = p_t$.

(b) Conclude that both e and p satisfy the wave equation.

11. Suppose $u(x,t)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$. Show that

   (a) For each $y \in \mathbb{R}$, the function $u(x-y,t)$ also satisfies the above equation.

   (b) Both u_x and u_t satisfy the above equation.

   (c) For any $a > 0$, the function $u(ax, at)$ satisfies the above equation. Note that the restriction $a > 0$ is not necessary.

12. (a) Let $u(x,t)$ be a solution of the wave equation with $c = 1$, valid for all x, t. Prove that for all x, t, h, k

$$u(x+h, t+k) + u(x-h, t-k) = u(x+k, t+h) + u(x-k, t-h).$$

   (b) Write a corresponding identity if u satisfies the wave equation with $c = 2$.

13. Solve

$$u_{tt} = 9u_{xx}$$

   where $u(x,0) = 1 - x^2$ and $u_t(x,0) = \cos(x)$ for all $x > 0$ and $u(0,t) = 0$ for all $t > 0$.

13′. (Old version) Consider the quarter-plane problem

$$\begin{aligned} u_{tt} &= 4u_{xx}, && x > 0,\ t > 0, \\ u(0,t) &= 0, && t > 0, \\ u(x,0) &= \phi(x), && x > 0, \\ u_t(x,0) &= \psi(x), && x > 0. \end{aligned}$$

   Let $\phi(x)$ be the function described in 9 and let $\psi(x) \equiv 0$. Sketch the solution $u(x,t)$ as a function of x for $t = \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, 1, 2$.

Homework 3 Solution

DongGyu Lim

1. Read the textbook to understand the situation well.

(a) u: the density function at the moment and v is the speed function.
If the density is high, people tend to reduce the speed, so v should be a decreasing function of u.

(b) The maximum velocity is obtained when there is no cars nearby: the density $=0$.
When the density is maximum, cars will stop moving: the speed is 0.
Mathematically, $v_{max} = v(0)$ and u_{max} is a positive real number such that $v(u_{max}) = 0$. (The notation for this is $u_{max} = \underset{u}{\operatorname{argmax}}\, v(u)$.)

(c) $Q(u) = u \cdot v(u)$ is a nonnegative continuous function. Moreover, it obtains zeros at $u=0$ and $u = u_{max}$. Hence, the image of the closed interval $[0, u_{max}]$ under the function Q should be a closed and bounded set. (Here, we use the fact that the image of a compact set under a continuous map is compact. We also use the fact that, in $\mathbb{R}$, being compact is equivalent to being closed and bounded.) We know that $Q(\frac{u_{max}}{2}) > 0$, so the image of $[0, u_{max}]$ has a maximum larger than zero. This means that there exists $u_0 \in (0, u_{max})$ s.t. $Q(u_0)$ is the maximum.

(d) Yes, you can use the hint. Choose $v(u)$ as a cubic polynomial of u carefully. One condition you can easily recognize: $v(u) = 0$ should have exactly one root at u_{max}. By scaling, one can assume $v(u)$ is $(1-u) \cdot f(u)$ where f is a quadratic polynomial without zero, that is, it is of the form $(u-\alpha)^2 + \beta$ for some $\alpha \in \mathbb{R}$ and $\beta > 0$. Carefully considering that $v(u)$ is decreasing and $u \cdot v(u)$ should have two local maxima, you can find an example. For example, $v(u) = (1-u)\cdot(u^2 - u + \frac{1}{3})$.

⚡ An example for part d is not easy nor trivial to find. Some "references" say that 'you can find a function $f(t)$ obviously' ($f(t)$: the one appearing in my solution) but this takes quite long time. So, you should not just claim without any specific example.

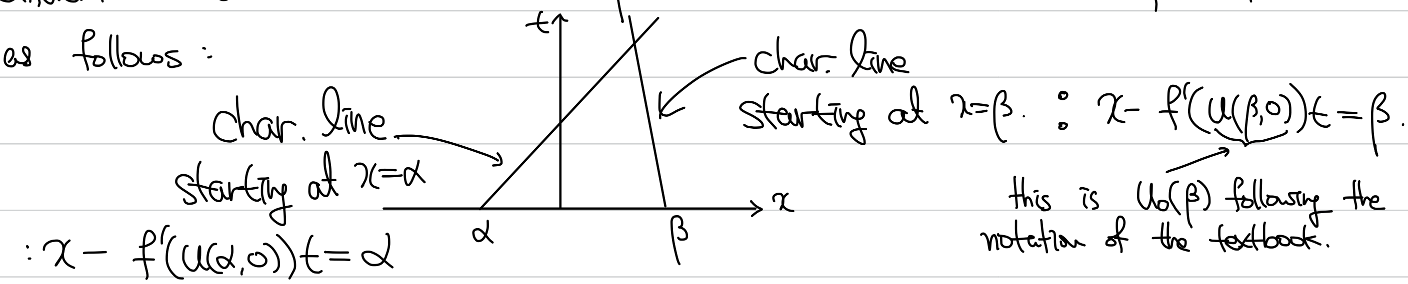
Thank you Antonio for suggesting another proof of part c.

Dong Gyu Lim

2. Idea: It would be better to think about this problem using problem 3 and 4. For example, if you choose $f(u) = \frac{1}{2}u^2$, then you will see the same equation as problem 4. ($f(u) = u - u^2 \Rightarrow$ problem 3.) Also, please read 3.4 of textbook carefully.

You can almost copy the proof of the textbook. One thing you should observe is that $f(u)_x$ is $f'(u) \cdot u_x$ by Chain Rule. Then the differential equations in the book becomes $\frac{dx}{dt} = f'(u)$, $\frac{du}{dt} = 0$. Imposing an initial condition, you end up getting $u = u_0(x - f'(u) \cdot t)$. This is the implicit equation.

For the solution to remain smooth, the characteristic lines should not intersect at any (positive) time. If any of two characteristic lines intersect, that will be the moment the solution breaks down because it would obtain (more than) two different values at that intersection point. If we draw the xt-plane, it becomes as follows:



Hence, the intersection point can be computed from $\alpha + f'(u_0(\alpha))t = \beta + f'(u_0(\beta))t$.
This gives you
$$t = \frac{\beta - \alpha}{f'(u_0(\alpha)) - f'(u_0(\beta))}.$$
Without loss of generality, we may assume $\alpha < \beta$. Then, $t > 0 \iff (f' \circ u_0)(\alpha) > (f' \circ u_0)(\beta)$.
Therefore, if we need the solution to remain smooth for all time, $f' \circ u_0$ should be a monotonically increasing function.

If this is violated, the minimum value of $\frac{\beta - \alpha}{f'(u_0(\alpha)) - f'(u_0(\beta))}$ should be the time the solution breaks down. Using the mean value theorem, we know that there exists some $\gamma \in (\alpha, \beta)$ s.t $\frac{(f' \circ u_0)(\alpha) - (f' \circ u_0)(\beta)}{\alpha - \beta} = (f' \circ u_0)'(\gamma) = (f''(u_0) \cdot u_0')(\gamma)$. Therefore, minimal time t is $-\frac{1}{f''(u_0(\gamma)) \cdot u_0'(\gamma)}$ for $\gamma \in \mathbb{R}$ s.t $|f''(u_0(\gamma)) \cdot u_0'(\gamma)|$ is minimum and $f''(u_0(\gamma)) \cdot u_0'(\gamma) < 0$.

Well... we actually need to assume that u_0 is smooth b/c even at $t=0$ we need smoothness.

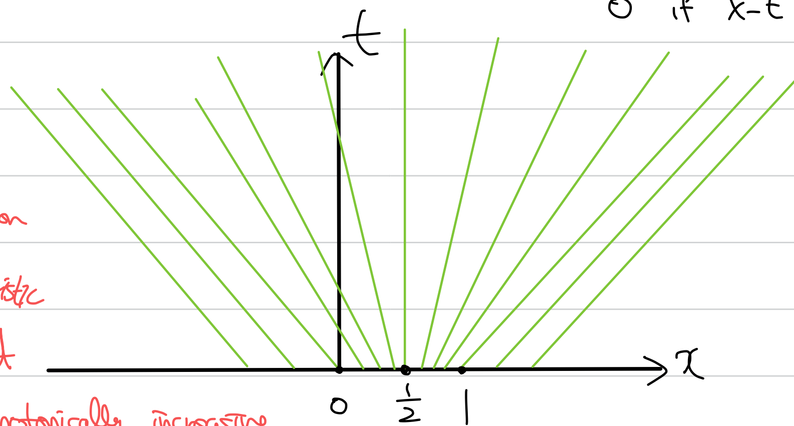
DongGyu Lim

3. Idea: In problem 2, we already found the way this problem works : $u(x,t)$ is constant along the characteristic lines.

We use the method of characteristics : $\frac{dx}{dt} = 1-2u$ and $\frac{du}{dt} = 0$ and $u_0(x) = \begin{cases} 1 & x<0 \\ 1-x & x\in(0,1) \\ 0 & x>1 \end{cases}$.

If we start from $x_0<0$, $u=1$ and $x = -t + x_0$.
$x_0 \in (0,1)$, $u = 1-x_0$ and $x = (2x_0-1)t + x_0$. So, $u(x,t) = \begin{cases} 1 & \text{if } x+t<0. \\ \frac{1-x}{2t+1} & \text{otherwise.} \\ 0 & \text{if } x-t>1. \end{cases}$
$x_0 > 1$, $u = 0$ and $x = t + x_0$

The family of characteristics can be drawn as :



ϟ u_0 is not smooth, however the solution exists for all time because the characteristic lines do not intersect. Considering the result of problem 2, this is because $1-2u_0$ is monotonically increasing.

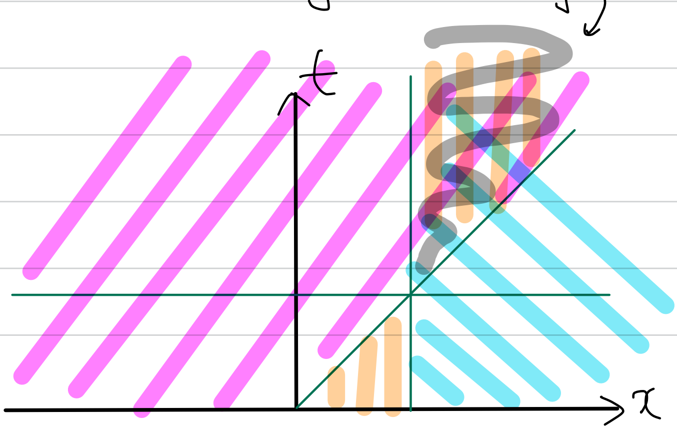
4. Almost the same solution as problem 3, but here the solution will break down because u_0 is NOT monotonically increasing.

Similar to problem 3, we get $x_0<0$ $u=1$ and $x = t+x_0$
$x_0\in(0,1)$ $u = 1-x_0$ and $x = (1-x_0)t + x_0$.
$x_0>1$ $u=0$ and $x = x_0$

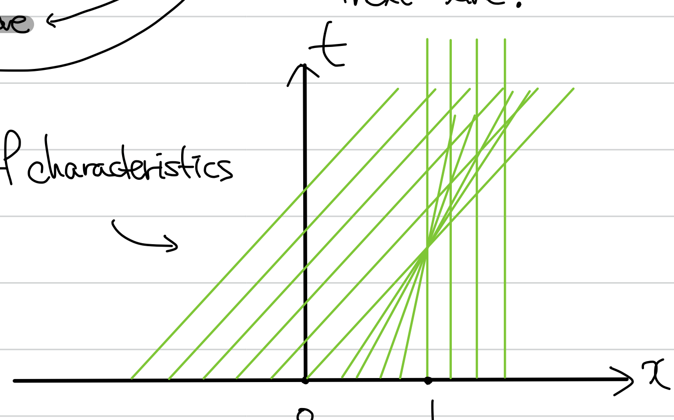
$\therefore u(x,t) = \begin{cases} 1 & \text{if } x-t<0 \quad \text{without } x>1 \\ \frac{x-1}{t-1} & \text{if } 0<\frac{t-x}{t-1}<1 \iff t<x<1 \text{ OR } 1<x<t \\ 0 & \text{if } x>1 \quad \text{without } t>x \end{cases}$ remove

these gray conditions (are coming from the next line.)

Now, we need to remove the cases where the regions are intersecting.



family of characteristics



ϟ For now, you can draw all the lines for all time t. But, you can think about where the lines should stop actually.

DongGyu Lim

5. Idea: Apply d'Alembert's solution.

The solution for $u_{tt} = c^2 u_{xx}$ with $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ is

$$\frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy$$

This is called d'Alembert's solution. (See 4.2 of the textbook.)

Therefore, $u(x,t) = \frac{1}{2}\left(e^{x+ct} + e^{x-ct}\right) + \frac{1}{2c}\left(\cos(x-ct) - \cos(x+ct)\right)$.

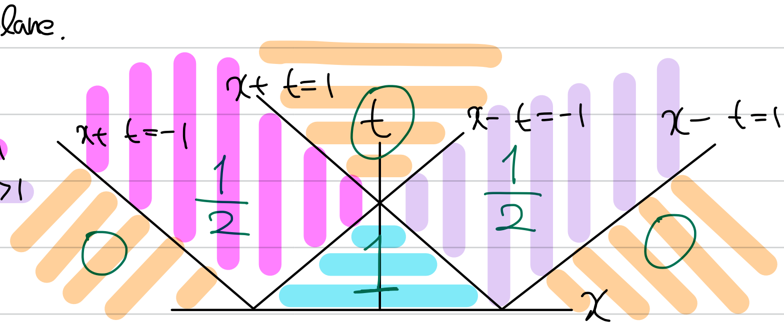
ϟ I recommend you to try to induce d'Alembert's solution from the fact that $u(x,t) = F(x+ct) + G(x-ct)$ for some F & G. It is not difficult at all nor time-consuming, but it will be helpful for memorizing the formula.

6. SAME AS BEFORE.

$u(x,t) = \frac{1}{2}\left[\phi(x+t) + \phi(x-t)\right]$ where $\phi(x) = \mathbb{1}_{(-1,1)}$. (1 only on $(-1,1)$).

It would be better to draw xt-plane.

$$u(x,t) = \begin{cases} 1 & \text{if } x+t<1 \text{ and } x-t>-1 \\ \frac{1}{2} & \text{if } -1<x+t<1 \text{ and } x-t<-1 \\ & \text{OR } -1<x-t<1 \text{ and } x+t>1 \\ 0 & \text{otherwise.} \end{cases}$$



7. Idea: d'Alembert's solution

$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy$. To see if this is even or odd or neither with respect to x (not w.r.t t), we need to consider

$$\begin{aligned} u(-x,t) &= \frac{1}{2}\left[\phi(-x+ct) + \phi(-x-ct)\right] + \frac{1}{2c}\int_{-x-ct}^{-x+ct}\psi(y)\,dy. \\ &= \frac{1}{2}\left[-\phi(x-ct) - \phi(x+ct)\right] + \frac{1}{2c}\int_{x+ct}^{x-ct}\psi(-y)\,d(-y) \\ &= -\frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\underbrace{\psi(-y)}_{=-\psi(y)}\,dy \\ &= -\left(\frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy\right) \end{aligned}$$

because ψ is odd.

Therefore, $u(-x,t) = -u(x,t)$. So, it is odd w.r.t x for all time t.

DongGyu Lim

8. Idea: "A hammer whose head diameter is 2a" & "the middle is hit by the hammer" means that at $t=0$, $\phi(x)$ (the initial function) is zero outside the range of the head and there is some nonzero function in the range.

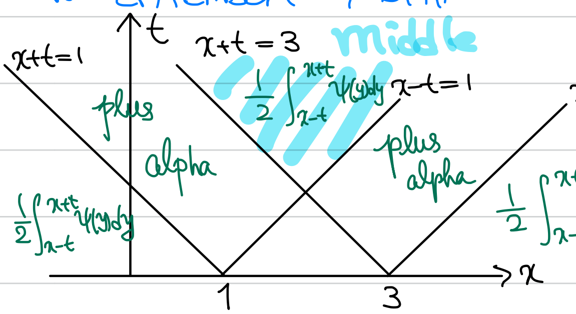
$u_{tt} = \frac{T}{\rho} u_{xx}$ and $u(x,0) = \phi(x)$ where $\phi(x)$ has the property above & $u_t(x,0) \equiv 0$ is the equation!

Here, we are regarding the midpoint as $x=0$. So, $u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)]$ where $c = \sqrt{\frac{T}{\rho}}$.

Therefore, if the fly is sitting on $x = l/2$, it will be affected when the time satisfies

$a = \frac{l}{2} - ct \Rightarrow t = \frac{l-2a}{2c} = \frac{1}{2}\sqrt{\frac{\rho}{T}}(l-2a)$.

9. d'Alembert AGAIN.



$c=1$ in this problem. The middle section is where $x+t>3$ and $x-t<1$. Therefore, $u(x,t) = \frac{1}{2}[\phi(x+t)+\phi(x-t)] + \frac{1}{2}\int_{x-t}^{x+t}\psi(y)dy$

is $\frac{1}{2}[0+0] + \frac{1}{2}\int_{<1}^{>3}\psi(y)dy = \frac{1}{2}\int_1^3 \psi(y)dy$

because $\psi(y) = 0$ outside $[1,3]$.

Hence, in the middle section, it is constant $\frac{1}{2}\int_1^3\psi(y)dy$. The necessary and sufficient condition is that $\int_1^3 \psi(y)dy = 0$.

↯ There is an insignificant typo in the problem: "middle section with $t > \frac{1}{2}$" ⟶ "middle section with $t>1$"

10. Just do it!

(a) $e_t = \left(\frac{1}{2}(u_t^2+u_x^2)\right)_t = \frac{1}{2}\cdot 2u_t \cdot u_{tt} + \frac{1}{2} 2u_x \cdot u_{xt} = u_t u_{tt} + u_x u_{xt}$.

$p_x = (u_t u_x)_x = u_{tx}\cdot u_x + u_t u_{xx}$. Comparing them by using $u_{tt} = u_{xx}$, we get the same terms. Note also that u is C^3, so $u_{tx} = u_{xt}$ by Clairaut's theorem.

$e_x = u_t \cdot u_{tx} + u_x \cdot u_{xx}$ & $p_t = u_{tt} u_x + u_t \cdot u_{xt}$ ⟹ Clairaut's theorem & $u_{tt} = u_{xx}$ imply that $e_x = p_t$ as well.

(b) $e_t = p_x \Rightarrow e_{tt} = p_{xt}$
$e_x = p_t \Rightarrow e_{xx} = p_{tx}$ } the same because p is C^2 (u is C^3 & $p = u_t u_x$). $\therefore e_{tt} = e_{xx}$.

You can easily verify p also satisfies the equation.

Dongyu Lim

11. Follow the instruction! (I believe) When the book says equation 4.25 they assume that the solution u is C^3.

(a) Let's define $v(x,t)$ to be $u(x-y,t)$. Then, $v_x(x,t) = 1\cdot u_x(x-y,t)$ and $v_{xx}(x,t) = 1\cdot 1\cdot u_{xx}(x-y,t)$. Similarly, one finds $v_t(x,t) = u_t(x-y,t)$ and $v_{tt}(x,t) = u_{tt}(x-y,t)$.
The wave equation is satisfied at any points (x,t), so $u_{tt}(x-y,t) = c^2\cdot u_{xx}(x-y,t)$. Therefore, $v_{tt} = c^2 v_{xx}$.

(b) Note that we are assuming that u is C^3 so that we can consider u_{xtt} kind of functions. Moreover, we can change the order of derivation up to 3^{rd} derivatives, for example, $u_{xxt} = u_{xtx} = u_{txx}$. (Just to make clear, the following is not true: $u_{xxtt} = u_{ttxx}$.)
Now, everything becomes clear because $(u_x)_{tt} = u_{xtt} = u_{ttx} = (u_{tt})_x = (c^2\cdot u_{xx})_x = c^2\cdot(u_x)_{xx}$.
Similarly, we have $(u_t)_{tt} = u_{ttt} = (u_{tt})_t = c^2\cdot(u_{xx})_t = c^2\cdot u_{xxt} = c^2 u_{txx} = c^2\cdot(u_t)_{xx}$.

(c) Let $v(x,t)$ be $u(ax,at)$. Then, $v_x(x,t) = a\cdot u_x(ax,at)$ and $v_{xx}(x,t) = a\cdot a\cdot u_{xx}(ax,at)$.
On the other hand, $v_t(x,t) = a\cdot u_t(ax,at)$ and $v_{tt}(x,t) = a\cdot a\cdot u_{tt}(ax,at)$. Again, the wave equation is satisfied at any points, $u_{tt}(ax,at) = c^2\cdot u_{xx}(ax,at) \Rightarrow v_{tt} = c^2 v_{xx}$. Note that we did not use the assumption that $a>0$.

12. Idea: Use the general solution $F(x+ct) + G(x-ct)$. You could use the d'Alembert's solution, but then you need to write down a bunch of integral signs.

(a) Let $u(x,t)$ be $F(x+t) + G(x-t)$ for some F and G.

$$u(x+h,t+k) - u(x+k,t+h) = G(x-t+h-k) - G(x-t-(h-k)) \quad \cdots (1)$$
$$-h, \; -k \qquad\quad -k, \; -h \quad = \quad -(h-k) \qquad\quad +(h-k) \quad \cdots (2)$$

(1) + (2) gives you that the left hand sides add up to zero and it is exactly what we want: $u(x+h,t+k) + u(x-h,t-k) = u(x+k,t+h) + u(x-k,t-h)$.

(b) If u is a solution of $u_{tt} = 2^2 u_{xx}$, then $v(x,t) := u(x,\frac{t}{2})$ satisfies $v_{tt} = v_{xx}$, so it will satisfy the equation from part a. Writing that down, we have

$$v(x+h,t+k) + v(x-h,t-k) = v(x+k,t+h) + v(x-k,t-h).$$
$$\therefore \; u\left(x+h,\tfrac{t+k}{2}\right) + u\left(x-h,\tfrac{t-k}{2}\right) = u\left(x+k,\tfrac{t+h}{2}\right) + u\left(x-k,\tfrac{t-h}{2}\right).$$

To make the equation a bit good-looking, we can replace t by $2t$ and k,h by $2k, 2h$.

$$\therefore \; u(x+2h,t+k) + u(x-2h,t-k) = u(x+2k,t+h) + u(x-2k,t-h).$$

DongGyu Lim

13. Idea: Even in the quarter-plane problem, we have $u(x,t) = F(x+ct) + G(x-ct)$.

Check (4.17) from the textbook.

We have $u(x,t) = \begin{cases} \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy & \text{if } x-ct>0 \\ \frac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)dy & \text{otherwise.} \end{cases}$

In our situation $c=3$ and $\phi(x) = 1-x^2$ and $\psi(x) = \cos x$.

Therefore, $u(x,t) = \begin{cases} \frac{1}{2}\left((x+3t)^2 + (x-3t)^2\right) + \frac{1}{6}\int_{x-3t}^{x+3t} \cos y \, dy & \text{if } x>3t \\ \frac{1}{2}\left((x+3t)^2 - (3t-x)^2\right) - \frac{1}{6}\int_{3t-x}^{3t+x} \cos y \, dy & \text{if } 0<3t-x \end{cases}$

Making them simpler, one gets $u(x,t) = \begin{cases} x^2 + 9t^2 + \frac{1}{6}\left(\sin(x+3t) - \sin(x-3t)\right) & \text{if } x>3t. \\ 6xt - \frac{1}{6}\left(\sin(x+3t) - \sin(3t-x)\right) & \text{if } 0<x<3t. \end{cases}$

I accidentally did the homework of former version. Here is the solution for the previous version's #13. (= Shearer & Levy 4.5)

Dongyu Lim

13'. Idea: Even in the quarter-plane problem, we have $u(x,t) = F(x+ct) + G(x-ct)$.

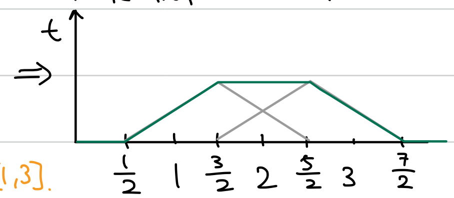
Check (4.17) from the textbook.

We have
$$u(x,t) = \begin{cases} \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy & \text{if } x-ct>0 \\ \frac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x}\psi(y)\,dy & \text{otherwise.} \end{cases}$$

Fortunately, we have $\psi \equiv 0$. So, it is much better. Note that $c=2$.

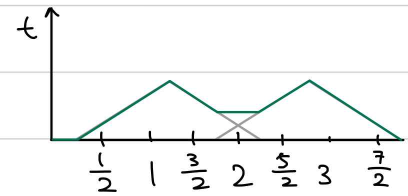
$$t=\tfrac{1}{4}: \; u(x,\tfrac{1}{4}) = \begin{cases} \frac{1}{2}\left[\phi(x+\frac{1}{2}) + \phi(x-\frac{1}{2})\right] & \frac{1}{2}<x \\ \frac{1}{2}\left[\phi(x+\frac{1}{2}) - \underbrace{\phi(\tfrac{1}{2}-x)}\right] & x<\frac{1}{2} \end{cases} \Longrightarrow$$

Zero as $\operatorname{supp}\phi = [1,3]$.



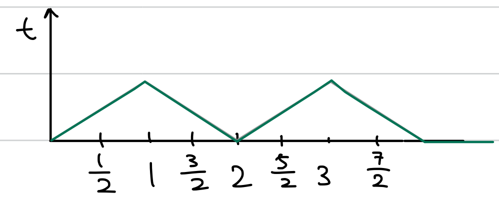
$$t=\tfrac{3}{8}: \; u(x,\tfrac{3}{8}) = \begin{cases} \frac{1}{2}\left[\phi(x+\frac{3}{4}) + \phi(x-\frac{3}{4})\right] & x>\frac{3}{4} \\ \frac{1}{2}\left[\phi(x+\frac{3}{4}) - \underbrace{\phi(\tfrac{3}{4}-x)}\right] & x<\frac{3}{4} \end{cases} \Longrightarrow$$

Zero as $\operatorname{supp}\phi = [1,3]$.



$$t=\tfrac{1}{2}: \; u(x,\tfrac{1}{2}) = \begin{cases} \frac{1}{2}\left[\phi(x+1) + \phi(x-1)\right] & x>1 \\ \frac{1}{2}\left[\phi(x+1) - \underbrace{\phi(1-x)}\right] & x<1 \end{cases} \Longrightarrow$$

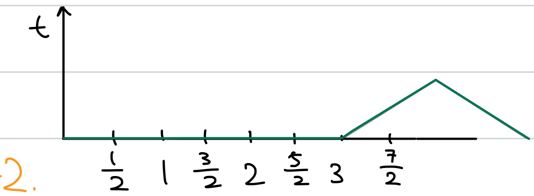
Zero as $\operatorname{supp}\phi = [1,3]$.



Zero because $\operatorname{supp}\phi = [1,3]$.

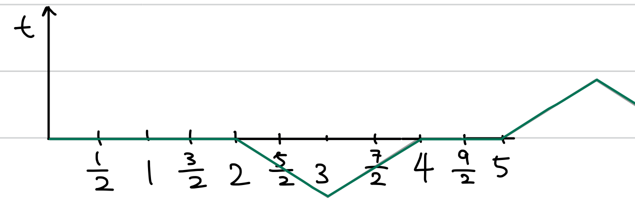
$$t=1: \; u(x,1) = \begin{cases} \frac{1}{2}\left[\underbrace{\phi(x+2)} + \phi(x-2)\right] & x>2 \\ \underbrace{\frac{1}{2}\left[\phi(x+2) - \phi(2-x)\right]} & x<2 \end{cases}$$

Zero because ϕ is symmetric w.r.t $x=2$.



$$t=2: \; u(x,2) = \begin{cases} \frac{1}{2}\left[\underbrace{\phi(x+4)} + \phi(x-4)\right] & x>4 \\ \frac{1}{2}\left[\phi(x+4) - \phi(4-x)\right] & x<4 \end{cases}$$

Zero because $\operatorname{supp}\phi = [1,3]$.



↳ I strongly recommend you to read 4.2.2 and carefully think about another perspective that is related to odd functions: "oddify" your $\phi(x)$ to the left half of the real line (x-axis).

BONUS