Homework 2 - Spring 2020 - MATH 126-001 - Introduction to PDEs

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1. Show the system of equations

$$x^2 - y^2 - u^3 + v^2 + 4 = 0$$
$$2xy + y^2 - 2u^2 + 3v^4 + 8 = 0$$

can express solutions of the form $u = g_1(x, y)$, $v = g_2(x, y)$ near $x = 2$ and $y = -1$ such that

$$g_1(2, -1) = 2$$
$$g_2(2, -1) = 1.$$

Solution: We first check that the point $(x, y, u, v) = (2, -1, 2, 1)$ is a solution to the above equations. Indeed

$$4 - 1 - 8 + 1 + 4 = 0$$
$$-4 + 1 - 8 + 3 + 8 = 0$$

Now, we use the implicit function theorem.

> Let me be precise here. I would like to use 'Implicit Function Theorem III' in the lecture note. We have $A = \mathbb{R}^2 \times \mathbb{R}^2$ and $\mathbf{x}_0 = (2, -1)$ and $\mathbf{u}_0 = (2, 1)$. Now, the function defined by $\mathbf{F}(\mathbf{x}, \mathbf{u}) := (x^2 - y^2 - u^3 + v^2 + 4, 2xy + y^2 - 2u^2 + 3v^4 + 8)$ where $\mathbf{x} = (x, y)$ and $\mathbf{u} = (u, v)$ satisfies $\mathbf{F}(\mathbf{x}_0, \mathbf{u}_0) = (0, 0)$ as we just checked. So, now we need to find J and check invertibility.

We need to check that the following matrix of partial derivatives is invertible at $(2, -1, 2, 1)$:

$$\begin{pmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{pmatrix}$$

Plugging in, we get:

$$\begin{pmatrix} -12 & 2 \\ -8 & 12 \end{pmatrix}$$

which has determinant -128 and so is invertible as desired. Hence, by the implicit function theorem we know that we can write $u = g_1(x, y)$ and $v = g_2(x, y)$ for some functions g_1, g_2 near $x = 2, y = -1$.

2. Suppose

$$y' = \frac{-t + \sqrt{t^2 + 4y}}{2}$$
$$y(2) = -1$$

a. Show

$$y_1 = 1 - t \text{ and } y_2 = -\frac{t^2}{4}$$

are solutions.

b. Do these two solutions violate the uniqueness theorem for the IVP?

Solution: We check that y_1 and y_2 satisfy the above differential equations:

$$\frac{-t+\sqrt{t^2+4(1-t)}}{2} = \frac{-t+(t-2)}{2} = -1 = y_1'$$
$$y_1(2) = -1$$

and

$$\frac{-t+\sqrt{t^2+4(\frac{-t^2}{4})}}{2} = \frac{-t+(t-2)}{2} = -\frac{t}{2} = y_2'$$
$$y_2(2) = -1$$

For (b), these solutions do not violate the uniqueness theorem because $f(t,y) = \frac{-t+\sqrt{t^2+4y}}{2}$ is not C^1. Indeed, we have

$$f_y = (t^2+4y)^{-\frac{1}{2}}$$

which is not continuous at $(2,-1)$. (It is not even defined.)

3. Use the substitution $v = u_y$ to solve for $u = u(x,y)$:

$$u_{xy} = 5u_y, \quad u(x,x) = 0, \quad u_y(x,x) = 2.$$

Solution:

Here is some legit concern: Is $u_{xy} = u_{yx}$ in this problem?

Personally, I think: When you feel like the point of the problem seems not something very profound or deep, you can assume all good things.

Using the substitution $v = u_y$, we can convert our PDE into a simpler one:

$$v_x = 5v.$$

Hence, $v(x,y) = c(y)e^{5x}$. One of the conditions says $v(x,x) = 2$. So, $c(x) = 2e^{-5x}$ and we get $v(x,y) = 2e^{5x}e^{-5y}$. Now, $u(x,y) = \int v(x,y)dy + \tilde{c}(x) = -\frac{2}{5}e^{5(x-y)} + \tilde{c}(x)$. However, we are given that $u(x,x) = 0$ so that $\tilde{c}(x) = \frac{2}{5}$. Therefore, the solution $u(x,y)$ is $\frac{2}{5}(1-e^{5(x-y)})$.

Please double check if your answer really satisfies the given equation (if time allows)!

4. Solve the initial value problems for $x \in \mathbb{R}$ and $t > 0$.

a. $u_t - 2u_x = 0$, where $u(x,0) = \sin(2x)$.

b. $u_t + 2u_x = 0$, where $u(0,t) = \cos(3t)$.

Solution: For part (a), we note that the function $f(x,t) = u(x-2t,t)$ satisfies $f_t = -2u_x + u_t = 0$. In particular, f does not depend on t. This implies that $u(x,t) = g(x+2t)$ for some function g. Since $u(x,0) = \sin(2x)$, we see that $u(x,t) = \sin(2x+4t)$ is the solution.

Please double check if your answer really satisfies the given equation (if time allows)!

For part (b), we make an analogous analysis. We have $f(x,t) = u(x+2t,t)$ satisfies $f_t = 2u_x + u_t = 0$ so that $u(x,t) = g(x-2t)$ for some function g. Since $g(-2t) = u(0,t) = \cos(3t)$, we have $u(x,t) = g(x-2t) = \cos(-\frac{3(x-2t)}{2})$ is the solution.

Please double check if your answer really satisfies the given equation (if time allows)!

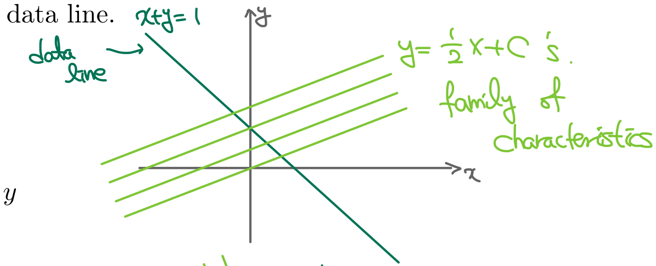
5. Use the method of characteristics to solve the problem

$$u_y(x,y) + cu_x(x,y) = q(x,y,u)$$
$$u(x,y) = f(x,y)$$

on the data line $ax+by+d=0$ with constant c, functions q and f and line $ax+by+d=0$ as indicated. In each case, sketch the family of characteristics in the (x,y)-plane and the data line.

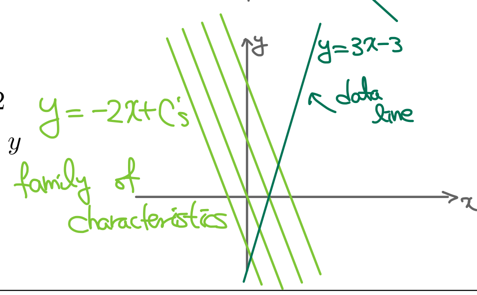
a. Let

$$c = 2$$
$$q(x,y,u) = u$$
$$f(x,y) = 2x - y$$
$$x + y - 1 = 0$$



b. Let

$$c = -1/2$$
$$q(x,y,u) = 2x + y$$
$$f(x,y) = e^{x-y}$$
$$3x - y - 3 = 0$$



Solution: We do part (a) first. First of all, we parameterize the initial condition as follows: $(x,y,u) = (r, 1-r, 3r-1)$. The characteristic curves will satisfy the following system of ODEs

$$\frac{dx}{ds} = 2$$
$$\frac{dy}{ds} = 1$$
$$\frac{du}{ds} = u$$

These give the equation $x = 2s + c_1(r), y = s + c_2(r), u = c_3(r)\cdot e^s$. With the given initial condition, this will be uniquely decided. Here, you just find what c_1, c_2, and c_3 are by looking at the initial condition at $s = 0$.

We get $c_1 = r$, $c_2 = 1-r$ and $c_3 = (3r-1)e^s$. Therefore, $x(r,s) = 2s+r$, $y(r,s) = s+1-r$, and $u(r,s) = (3r-1)e^s$. As we are finding u in terms of x and y, we need to convert r and s into x and y.

This can be done by just removing one of r and s (alternatively) and express them as a function of x and y. (Conceptually, we are explicitly finding inverse functions.)

We have $x + y = 3s + 1$ so that $s = \frac{x+y-1}{3}$ and $x - 2y = 3r - 2$ so that $r = \frac{x-2y+2}{3}$. Now, $u(x,y) = (3r-1)e^s$ becomes $(x - 2y + 1)e^{\frac{x+y-1}{3}}$.

Please double check if your answer really satisfies the given equation (if time allows)!

Similarly, we can do part (b). The initial condition is parameterized as $(r+1, 3r, e^{1-2r})$. The differential equation we need to solve is $x_s = -\frac{1}{2}$, $y_s = 1$, and $u_s = 2x + y$. (I am using f_s instead of $\frac{\partial f}{\partial s}$ for the sake of simplicity.)

Considering the initial condition, we get $x = -\frac{1}{2}s + r + 1$, $y = s + 3r$, and $u = (5r+2)s + e^{1-2r}$. Then, $2x + y = 5r + 2$ so that $r = \frac{2x+y-2}{5}$ and $3x - y = 3 - \frac{5}{2}s$ so that $s = \frac{-6x+2y+6}{5}$.

Hence, $u(x,y) = \frac{(2x+y)(-6x+2y+6)}{5} + e^{\frac{9-4x-2y}{5}}$.

Please double check if your answer really satisfies the given equation (if time allows)!

6. Solve the IVP

$$u_t(x,t) + u(x,t)u_x(x,t) = 2t, t > 0, x \in \mathbb{R}$$
$$u(x,0) = x, x \in \mathbb{R}$$

Solution: The strategy is the same as before: we first need to find the initial condition and then solve some a system of differential equations with only one variable derivatives.

$u(x,0) = x$ can be parameterized as $(r, 0, r)$, $r \in \mathbb{R}$. Now, the system of differential equations we are to solve is

$$\frac{dx}{ds} = u$$
$$\frac{dt}{ds} = 1$$
$$\frac{du}{ds} = 2t$$

From the second equation and the initial condition, we have $t = s$. Plugging in to the third one and considering the initial condition, we have $u = s^2 + r$. Finally, the first one gives $x = \frac{1}{3}s^3 + rs + r$.

Our goal is to express u in terms of x and t, we need to find r and s in terms of x and t. Fortunately, s is simple because it is just t. For r, we will plug in $s = t$ to $x = \frac{1}{3}s^3 + rs + r$. Then, we get $rt + r = x - \frac{1}{3}t^3$, so $r = \frac{x - \frac{1}{3}t^3}{t+1}$.

Therefore, $u(x,t) = t^2 + \frac{x - \frac{1}{3}t^3}{t+1} = \frac{3x + 2t^3 + 3t^2}{3(t+1)}$.

Please double check if your answer really satisfies the given equation (if time allows)!

7. Solve the IVP

$$\sqrt{x}u_x - \sqrt{y}u_y = \frac{u}{\sqrt{x} - \sqrt{y}}$$
$$u(4x, x) = \sqrt{x}$$

Solution: Now, you know what to do:

$$\frac{dx}{ds} = \sqrt{x}$$
$$\frac{dt}{ds} = -\sqrt{y}$$
$$\frac{du}{ds} = \frac{u}{\sqrt{x} - \sqrt{y}}$$

However, this problem has somewhat subtle issue. (Presumably, this would not be the very same problem as the previous ones.) In this problem, you need to be careful about the region over which you are finding the solution. If your region contains points satisfying $\sqrt{x} = \sqrt{y}$, the equation will not be acceptable. Also, the question uses $\sqrt{x}$ and $\sqrt{y}$, so we really want the region x, $y \geqslant 0$.

If you draw the region, it is actually not connected (in some sense) because you have the first quadrant without $y = x$ part. So, your solution should be found over one of the two pieces divided by $y = x$. Now, you look at the initial condition: it is over the $y = \frac{1}{4}x$ line. Hence, we can expect our solution only be found over the region defined by $x \geqslant 0$, $y \geqslant 0$, and $y < x$. Keeping this in mind, we can proceed.

We parameterize the initial condition by $(4r^2, r^2, r)$ because we prefer to have r^2 instead of $\sqrt{r}$. Here, we need to impose the condition $r > 0$. (Think about the region and remember that a square-root is not negative.)

Going back to the system of equations, we have to solve an equation of the form

$$f' = c\sqrt{f}.$$

You can divide it by $\sqrt{f}$ and consider $(2\sqrt{f})' = f' \cdot 2\frac{1}{2\sqrt{f}} = \frac{f'}{\sqrt{f}}$.

Now, $2\sqrt{x} = s + c_1(r)$, $2\sqrt{y} = -s + c_2(r)$. Recalling the initial condition, we have $2\sqrt{4r^2} = c_1(r)$ and $2\sqrt{r^2} = c_2(r)$. Hence, $\sqrt{x} = \frac{s}{2} + 2r$, $\sqrt{y} = -\frac{s}{2} + r$. Again, considering the region over which our function will be defined, we need to impose the condition that $-2r < \frac{s}{2} < r$ and $s > -r$. Together, it is $-r < s < r$.

Plugging in, we get $u_s = \frac{u}{s+r}$. Re-writing the equation, we have $\frac{1}{u}du = \frac{1}{s+r}ds$ so that $\ln|u| = \ln|s+r| +$ function of r. We have $u = c(r) \cdot (s+r)$. Considering the initial condition, we get $c(r) = 1$ so that $u = s + r$.

Finally, we express r and s in terms of x and y: $\sqrt{x} + \sqrt{y} = 3r$ and $\sqrt{x} - 2\sqrt{y} = \frac{3}{2}s$ so that $r = \frac{1}{3}(\sqrt{x} + \sqrt{y})$ and $s = \frac{2}{3}(\sqrt{x} - 2\sqrt{y})$. Therefore, $u(x, y) = s + r = \sqrt{x} - \sqrt{y}$ over the region $0 \leqslant y < x$.

Please double check if your answer really satisfies the given equation (if time allows)!

8. Consider the differential equation

$$au_x + bu_y = 0$$

where $a, b \in \mathbb{R}$. We know from lecture that

$$u(x, y) = f(bx - ay)$$

where f is any function of one variable, from the method of characteristics. You will find the solution using the following change of coordinates:

$$r = ax + by, \quad s = bx - ay.$$

a. The coordinate (r, s) is orthogonal. Along which axis is the function $u(x, y)$ constant?
b. Show $u_x = au_r + bu_s$ and $u_y = bu_r - au_s$.
c. From part (b), find u in terms of f evaluated at r and s.

Solution: Caution. This problem gives another way to prove that $u(x, y)$ is a function of $bx - ay$. So, you should not use that $u(x, y) = f(bx - ay)$ actually.

For part (a), we can recall the fact that $au_x + bu_y$ can be thought of as the (a, b)-directional derivative of $u(x, y)$. So, $au_x + bu_y = 0$ means that along the direction parallel to (a, b) the function is constant. Now, we need to figure out if it is parallel to r-axis or s-axis. Remember that x-axis is defined by $y = 0$ not $x = 0$. The line that passes through the origin with the direction (a, b) is $bx - ay = 0$. So, it is $ax + by = r$ axis.

For part (b), using Chain Rule, we get $u_x = u_r \cdot \frac{dr}{dx} + u_s \cdot \frac{ds}{dx} = u_r a + u_s b = au_r + bu_x$. Similarly, we obtain $u_y = bu_r - au_s$.

For part (c), we know that $au_x + bu_y = a(au_r + bu_s) + b(bu_r - au_s) = (a^2 + b^2)u_r$. Hence, the differential equation given is equivalent to $u_r = 0$. (Again, we should ignore the case like $a = b = 0$ because it makes the differential equation empty.) Hence, r-directional derivative of u is 0 so that u only depends on the value of s, that is, $u(x, y) = f(s) = f(bx - ay)$.

9. Let $u(x, t)$ measure the population of bacteria at position $x \in \mathbb{R}$ at time t. Assume that the bacteria moves at a constant velocity c in the positive x direction and decays at a constant rate r. If the initial population is $u(x, 0) = f(x)$, find $u(x, t)$.

Solution: 1st method. The PDE we want to solve is a transport style equation and can be written as

$$u_t + cu_x = -ru.$$

Now, this problem is almost the same problem as 5-a. One can find that $x = cs + r$, $t = s$, $u = f(r)e^{-rs}$ so that $u(x, t) = f(x - ct)e^{-rt}$.

2nd method. Imagine that we are observing some point whose initial location is x_0. We think the point is moving with velocity c as time passes. Now, the population of the bacteria at the point we are observing can be written as $u(ct + x_0, t)$ at time t. The decay condition tells us that the population that disappears as time passes is proportional (with ratio r) to the population at that point. Hence, $\frac{d}{dt}u(ct + x_0, t) = -r \cdot u(ct + x_0, t)$.

x_0 is independent from t, so $u(ct + x_0, t) = c_1(x_0)e^{-rt}$. Now, using the initial condition, we can find that $c_1(x_0) = f(x_0)$ (by plugging in $t = 0$). Therefore, $u(ct + x_0, t) = f(x_0)e^{-rt}$ so that $u(x, t) = f(x - ct)e^{-rt}$.