Homework 1 - Spring 2020 MATH 126-001 - Introduction to PDEs

1. Show that there exists a continuous function that does not map open sets to open sets.

2. Show that there exists a continuous function that does not map simply connected sets to simply connected sets.

3. Let

$$I = \{(x, 0) \mid x \in (0, 1)\}$$
$$J = \{(x, 0) \mid x \in [0, 1]\}.$$

   (a) Is the set $\mathbf{R}^2 \backslash I$ open or closed?

   (b) Is the set $\mathbf{R}^2 \backslash J$ open or closed?

4. The velocity of a particle with respect to position is given by

$$v(x) = xe^x.$$

   Find the acceleration function with respect to position.

5. Let $f : \mathbf{R}^2 \to \mathbf{R}$ where $f \in C^2(\mathbf{R}^2)$. Let $s(\alpha, \beta) = \alpha + \beta t$ and suppose

$$G(a, b, c, d) = f(s(a, b), s(c, d)).$$

   Show that G satisfies the partial differential equation

$$\frac{\partial^2 G}{\partial c \partial b} = \frac{\partial^2 G}{\partial a \partial d}.$$

6. Suppose $y \in C^2((0, \infty))$ is a solution of the differential equation

$$t^2 y''(t) + \alpha t y'(t) + y(t) = 0.$$

   Let $x = \ln(t)$ and express the differential equation of $y(t)$ with respect to x.

7. Determine if the system

$$u(x, y, z) = x + xyz$$
$$v(x, y, z) = y + xy$$
$$w(x, y, z) = z + 2x + 3z^2$$

   can be solved for x, y, z in terms of (u, v, w) near $(x, y, z) = (0, 0, 0)$.

8. Define $\mathbf{F} : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$\mathbf{F}(x, y) = \left(e^x \cos(y), e^x \sin(y)\right).$$

   (a) Show that $\mathbf{F} \in C^1$ and that $d\mathbf{F}(x, y)$ is invertible for all $(x, y) \in \mathbf{R}^2$.

   (b) Is $\mathbf{F}$ one to one on all of $\mathbf{R}^2$? Is there a contradiction with the Inverse Function Theorem?

9. Consider the function

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

   (a) Show f' exists for all $x \in \mathbf{R}$.

   (b) Show $f'(0) > 0$.

   (c) Can the hypothesis that $f \in C^1$ be weakened in the Inverse Function Theorem?

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Homework 1 Solution

1. Idea : $f(x)=x$ or $f(x)=-x$ does not work.
Any strictly increasing or strictly decreasing function does not work.

$f(x)=\cos x$ for $x\in\mathbb{R}$ is a continuous function.
It sends $\mathbb{R}$ (open) onto $[-1,1]$ (not open).

[Another extreme example: $f(x)=0$ for $x\in\mathbb{R}$, a constant function]

2. Idea : The most famous non-simply connected set is the unit circle.

$f:\mathbb{R}\to\mathbb{R}^2$ (or one can restrict to $f:[0,2\pi]\to\mathbb{R}^2$).
defined by $f(x)=(\cos x, \sin x)$ is a continuous function.
It sends $\mathbb{R}$ or $[0,2\pi]$ (both simply connected) to the unit circle (non-simply connected).

3. Idea : "the complement of open is closed and vice versa".
"$\overline{C}=C$ implies C : closed" & "U : open means any point is in a small ball."

(a) $\mathbb{R}^2\setminus I$: closed or open? is the same question as
I : open or closed?
Is I open? No, because $(1/2,0)\in I$ does not have a small ball in I containing $(1/2,0)$.
closed? No, $\overline{I}$ contains $(0,0)$ which is not in I.
So, $\overline{I}\neq I$.

Therefore, $\mathbb{R}^2\setminus I$ is neither open nor closed.

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(b) Similarly to (a), we need to check about J. 2

J is not open for the same reason as I not open.

But, $\overline{J}$ is the same as J, so it is closed.

Therefore, $\mathbb{R}^2 \setminus J$ is not closed but it is open.

↯ Openness and Closedness are not exclusive.

Just like (a), a set can be neither of them.

Like (b), a set can be open but not closed.

Similarly, a set can be closed but not open.

Finally, just like $\mathbb{R}^2$, it can be both open and closed, but mostly there are only a few.

4. Idea: The acceleration is the derivative of the velocity with respect to the time. We need Chain rule.

Let t be the time variable and I will abuse notation so that $x(t)$ is the position "function" with respect to the variable t. Then, $v(x(t))$ is the velocity with respect to the time. So, the acceleration is $(v(x(t)))' = v'(x(t)) \cdot x'(t)$. However, $x'(t)$ is the velocity at time t which is just $v(x(t))$. Noting that $v'(x(t))$ means $\frac{dv}{dx}(x(t))$, the acceleration is $(x+1) \cdot e^x \cdot x \cdot e^x = x(x+1) \cdot e^{2x}$.

5. Idea: Use Chain rule to express the second partial derivatives of G in terms of f.

1) $$\frac{\partial^2 G}{\partial c \partial b} = \frac{\partial}{\partial c}\left(\frac{\partial G}{\partial b}\right) = \frac{\partial}{\partial c}\left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial b} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial b}\right)$$

Chain rule
$$= \frac{\partial}{\partial c}\left(\frac{\partial f}{\partial x}\cdot\frac{\partial s(a,b)}{\partial b} + \frac{\partial f}{\partial y}\cdot\frac{\partial s(c,d)}{\partial b}\right)$$

$$= \frac{\partial}{\partial c}\left(\frac{\partial f}{\partial x}\cdot t + \frac{\partial f}{\partial y}\cdot 0\right)$$

becomes $\frac{\partial}{\partial c}\left(\frac{\partial f}{\partial x}\cdot t\right) = \frac{\partial}{\partial c}\left(\frac{\partial f}{\partial x}\right)\cdot t + \frac{\partial f}{\partial x}\cdot \underbrace{\frac{\partial t}{\partial c}}_{0}$

prod. rule

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$$\overset{\text{chain rule}}{=} \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial x}{\partial c} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial y}{\partial c}\right]\cdot t$$

$$= \left[\frac{\partial^2 f}{\partial x^2}\cdot\underbrace{\frac{\partial s(a,b)}{\partial c}}_{=0} + \frac{\partial^2 f}{\partial y\partial x}\cdot\underbrace{\frac{\partial s(c,d)}{\partial c}}_{=1}\right] t = \frac{\partial^2 f}{\partial y\partial x}\times t.$$

2) In a similar way, you can compute $\frac{\partial^2 G}{\partial a\partial d}$

For the sake of simplicity, I'll use G_a instead of $\frac{\partial G}{\partial a}$ or f_y instead of $\partial f/\partial y$.

Now, $G_{da} = (G_d)_a = (f_x\cdot \underbrace{x_d}_{=0} + f_y\cdot \underbrace{y_d}_{=t})_a = (f_y\cdot t)_a$

$$\overset{\text{prod. rule}}{=} (f_y)_a\cdot t + f_y\cdot \underbrace{t_a}_{=0}$$

$$\overset{\text{chain rule}}{=} (f_{yx}\cdot \underbrace{x_a}_{=1} + f_{yy}\cdot \underbrace{y_a}_{=0})\cdot t$$

$$= f_{yx}\cdot t.$$

3) Now, we use Clairaut's theorem.

$\frac{\partial^2 G}{\partial c\partial b}$ is $\frac{\partial^2 f}{\partial y\partial x}\cdot t$ and $\frac{\partial^2 G}{\partial a\partial d}$ is $\frac{\partial^2 f}{\partial x\partial y}\cdot t$.

Once f is C^2, by Clairaut's theorem, they equal.

But this is given in the problem. So, they equal.

$(f\in C^2(\mathbb{R}^2))$

↳ You should end up getting f_{xy} and f_{yx} and need Clairaut's theorem to prove this problem. $f\in C^2(\mathbb{R}^2)$ condition is crucial.

6. Idea: Introduce a new function $z(x)$ defined by $y(e^x)$.

For example, if $y(t)$ is given as $y(t)=t^2$, then the new $z(x)$ is given as $z(x)=e^{2x}$.

We want to change y's diff. eqn with respect to t into z's diff. eqn with respect to x.

Note that $z(x)$ is defined to be $y(e^x)$.

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$$\frac{dz(x)}{dx} = \frac{d\,y(e^x)}{dx} = \frac{dy}{dt}(e^x)\cdot\frac{de^x}{dx} \quad \text{(Chain rule)}$$

$$= y'(e^x)\cdot e^x \quad \cdots\cdots (*)$$

(Note that $y'(e^x)$ is y' and then plug in e^x instead of t.)

$$\frac{d^2z(x)}{dx^2} = \frac{d}{dx}\left(y'(e^x)\cdot e^x\right) = \frac{d}{dx}\left(y'(e^x)\right)\cdot e^x + y'(e^x)\cdot\frac{de^x}{dx}$$

$$= \frac{dy'}{dt}(e^x)\cdot\frac{de^x}{dx}\cdot e^x + y'(e^x)\cdot e^x$$

$$= y''(e^x)\cdot e^{2x} + y'(e^x)\cdot e^x \quad \cdots\cdots (**)$$

If you replace t by e^x in the given differential equation, you get

$$e^{2x}\cdot y''(e^x) + \alpha\cdot e^x\cdot y'(e^x) + y(e^x) = 0.$$

Combining $(*)$ and $(**)$, we have

$$e^{2x}\cdot y''(e^x) = z''(x) - z'(x)$$

and $\quad e^x\cdot y'(e^x) = z'(x)$

and $\quad y(e^x) = z(x)$ (this is how we defined $z(x)$.)

So, we get a new differential equation

$$z''(x) - z'(x) + \alpha\cdot z'(x) + z(x) = 0$$

or, $\quad z''(x) + (\alpha - 1)\cdot z'(x) + z(x) = 0$

↳ $t^2\cdot y'' + \alpha t\cdot y' + y = 0$ is an example of Euler equations. What we have done here is to get an equivalent but easier-looking equation: $z'' + (\alpha-1)\cdot z' + z = 0$.

7. Idea: This is an Inverse Function Theorem problem. Recall what it is.

u, v, w are functions of x, y, z. The question is asking if x, y, z are functions of u, v, w near $(0,0,0)$.

According to Inverse Function Theorem, we only need to check if the Jacobian is invertible at $(x,y,z,u,v,w) = (0,0,0,0,0,0)$.

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It is $\begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} = \begin{pmatrix} 1+yz & xz & xy \\ y & 1+x & 0 \\ 2 & 0 & 1+6z \end{pmatrix}$. At $x=y=z=0$,

it becomes $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ ← lower triangular matrix whose determinant is $1 \times 1 \times 1 = 1$.

Therefore, the Jacobian is invertible. We know that u, v, w are all $C^1(\mathbb{R}^3)$, so by Inverse Function Theorem x, y, z can be expressed in terms of u, v, w.

8. Idea: Inverse Function Theorem asserts that there is an inverse function ONLY near the point.

(a) e^x and $\cos y$, $\sin y$ are all C^1 functions with respect to x and y. So, f sending (x,y) to $(e^x\cos y, e^x \sin y)$ is C^1. df is computed by $\begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{pmatrix}$ where $f=(f_1, f_2)$.

It is $\begin{pmatrix} e^x\cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$ and the determinant is $e^{2x}\cos^2 y - (-e^{2x}\sin^2 y) = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} > 0$.

So, df is invertible.

(b) f is not one-to-one on all of $\mathbb{R}^2$ because $f(0,0) = (1,0)$ and $f(0, 2\pi) = (1,0)$.

This does not contradict to Inverse Function Theorem because the theorem says f is one-to-one only near the point whose Jacobian is nonzero.

f is one-to-one in a small neighborhood of $(0,0)$ and so is in a small nbhd of $(0, 2\pi)$ separately.

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9. Idea: Follow the instruction...

(a) f' exists outside of $x=0$ for free (it is the product of differentiable functions).

Main thing is near $x=0$. To prove this, we need to check if $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ is well-defined. (In other words, if the limit exists)

$$\frac{f(x)-f(0)}{x-0} = \frac{\frac{x}{2}+x^2\sin\left(\frac{1}{x}\right)-0}{x} = \frac{1}{2} + x\sin\left(\frac{1}{x}\right).$$

As x goes to 0 (from right or left), x goes to 0 and $\sin\left(\frac{1}{x}\right)$ is always bounded by ± 1.

So, the limit exists and it is $\frac{1}{2}+0=\frac{1}{2}$.

(b) $f'(0)$ is computed in (a). It is $\frac{1}{2}>0$.

(c) $f(x)$ is differentiable as we have proven in (a), but it is not C^1 because $f'(x)$ is not continuous.

Why? $f'(x)$ outside $x=0$ can be computed using the product rule: $f'(x)=\frac{1}{2}+2x\sin\left(\frac{1}{x}\right)+x^2\cos\left(\frac{1}{x}\right)\cdot -\frac{1}{x^2}$

$$=\frac{1}{2}+2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right).$$

As x goes to 0, this oscillates radicall because of $\cos\left(\frac{1}{x}\right)$ term. So, the limit of $f'(x)$ as $x\to 0$ does not exist. It cannot be continuous at $x=0$.

Hence, one cannot use Inverse Function Theorem.

↳ It is not only that the theorem cannot be applied, but also that the theorem's statement does not hold. Even if $f'(0)>0$ and $f'(x)$ exists, f is not one-to-one in any small nbhd of $x=0$. You can be more convinced by considering:

- You can imagine the graph near 0. (It increases and decreases.)
- More concretely, $f'(x)$ is $-\frac{1}{2}<0$ if $x=\frac{1}{2n\pi}$ for any $n\in\mathbb{Z}_{>0}$.

  $\frac{3}{2}>0$ if $x=\frac{1}{(2n+1)\pi}$ for any $n\in\mathbb{Z}_{>0}$.

You can find large enough n s.t. $\frac{1}{2n\pi}, \frac{1}{(2n+1)\pi}$ is in your small nbhd of $x=0$.