Name: ______________________________ Student ID #: ________________________

This exam has 7 pages, 9 questions, and a total of **100** points.

If you are taking the class P/NP you may only complete the first 6 questions. If you are taking the class for a letter grade you may complete any of the questions.

1. I am taking the class for a letter grade:

   A. (0 points) Yes

   B. (30 points) No

2. (15 points) Find an entire function $f : \mathbf{C} \to \mathbf{C}$ such that

$$|f(3e^{it})| \leq 2$$

for all $t \in \mathbf{R}$ and

$$f(\sqrt{2} + i\sqrt{2}) = e$$

or state why no such function can exist. Make sure to justify your answer.

3. (15 points) The following came from a proof of Goursat's Theorem from complex analysis.

"Assume f is holomorphic on Ω and R is an open rectangle in Ω and $z_0 \in R \ldots$ from Cauchy's Theorem, we obtain

$$\left| \oint_{\partial R} f(z)\, dz \right| = \left| \oint_{\partial R} f(z) - f(z_0) - f'(z_0)(z - z_0)\, dz \right| \text{"}$$

Why can the author assume equality holds?

4. (15 points) Let $g : \mathbf{R}^2 \to \mathbf{R}$ be a continuous and bounded function and u be a C^∞-solution on $\mathbf{R}^2 \times (0,2)$ to the following heat equation:

$$u_t - (u_{xx} + u_{yy}) = u^3 \qquad \text{over } \mathbf{R}^2 \times (0,2)$$
$$u(x,0) = g(x) \qquad \text{for all } x \in \mathbf{R}^2$$

Moreover, suppose that u is bounded. Show that there exists a small enough $\epsilon > 0$ such that if $|g(x)|$ is bounded by ϵ for all $x \in \mathbf{R}$, then $|u(x,t)|$ is bounded by 2ϵ for all $(x,t) \in \mathbf{R}^2 \times (0,2)$. [Hint. Use Duhamel formula and bound $u(x,t)$.]

$|u(t,x)|$ is bounded by some number, say M, for all $(t,x) \in (0,2)\times\mathbb{R}^2$.
(We can choose $\varepsilon = \frac{1}{2}$. You'll see the reason at the end.)

Let T be the largest time in $[0,2]$ such that $|u(t,x)| \le 2\varepsilon$ for all $(t,x) \in [0,T]\times\mathbb{R}^2$. Applying Duhamel's formula,

$$u(t,x) = \underbrace{\int_{\mathbb{R}^2} \Phi(t, x-y)\, g(y)\, dy}_{\text{this is bounded by } \varepsilon} + \int_0^t \int_{\mathbb{R}^2} \Phi(t-s, x-y) \cdot \underbrace{u^3(s,y)}_{\le M^3}\, dy\, ds.$$

So, applying triangle inequality, we get

$$|u(t,x)| \le \varepsilon + t \cdot M^3. \quad \text{In other words, our } T \text{ is } \ge \frac{\varepsilon}{M^3} > 0.$$

Now, again by Duhamel's formula, we have $|u(T,x)| \le \varepsilon + (2\varepsilon)^3$
$\le \varepsilon + \frac{1}{2}\cdot\varepsilon$
if $\varepsilon \le \frac{1}{2}$.

The point is that we can apply Duhamel's formula again starting from $t = T$. Then, in a similar way, we get

$$|u(t,x)| \le \frac{3}{2}\varepsilon + (t-T)\cdot M^3$$

But, then this means that if t is slightly $\left(\frac{\varepsilon}{2M^3}\right)$ larger than T, $|u(t,x)|$ is still bounded by 2ε. This contradicts to the assumption that T is the largest one. So, there is no such $T \Rightarrow \forall \varepsilon \le \frac{1}{2}$, $|g(x)| \le \varepsilon$ implies $|u(t,x)| \le 2\varepsilon$ for all over $(0,2)\times\mathbb{R}^2$.

5. Let $u \in C^2(\Omega)$ where $\Omega = \mathbf{R} \times (0, \infty)$. Suppose u is a solution to the initial boundary value problem

$$\begin{aligned} u_t + u = u_{xx}, &\quad (x,t) \in \Omega \\ u(x,0) = g(x), &\quad x \in \mathbf{R} \end{aligned}$$

where g is integrable on $\mathbf{R}$.

(a) (10 points) Use the change of variables $u(x,t) = e^{-t}v(x,t)$ to express u in terms of the fundamental solution of the heat equation.

(b) (5 points) Suppose we have

$$\begin{aligned} u_t + f(t)u = u_{xx}, &\quad (x,t) \in \Omega \\ u(x,0) = g(x), &\quad x \in \mathbf{R} \end{aligned}$$

where g is integrable on $\mathbf{R}$.

What would be an appropriate change of variables to solve this IVP? You do not need to solve the problem, only state the change of variables.

6. Let $\Omega \subset \mathbf{R}^2$ be a simply connected, bounded domain, $u \in C^2(\Omega \times \mathbf{R})$, and $c : \Omega \times \mathbf{R} \to \mathbf{R}$ is bounded by $k \in \mathbf{R}$

$$|c(x, y, t)| \leq k, \quad (x, y) \in \Omega, \ t \geq 0.$$

Suppose u is a solution of

$$u_{tt} + c(x, y, t)u_t = \Delta u, \quad (x, y) \in \Omega, \ t > 0$$
$$u(x, y, t) = 0, \quad (x, y) \in \partial\Omega, \ t \geq 0.$$

Define the mathematical energy by

$$E(t) = \frac{1}{2} \iint_\Omega u_t^2 + |\nabla u|^2 \, dA.$$

(a) (5 points) Show

$$E'(t) \leq 2kE(t).$$

(b) (3 points) Show

$$\frac{d}{dt}\left(e^{-2kt}E(t)\right) \leq 0$$

for all $t \geq 0$.

$$\frac{d}{dt}\left(e^{-2kt}E(t)\right) = -2k \cdot e^{-2kt}E(t) + e^{-2kt} \cdot E'(t)$$
$$= \underbrace{e^{-2kt}}_{>0} \cdot \underbrace{\left(E'(t) - 2kE(t)\right)}_{\leq 0 \text{ by part a.}} \leq 0.$$

(c) (2 points) Suppose ~~$u(x, y, 0)$~~ $u_t(x,y,0) =$ $= 0$ for all $(x, y) \in \Omega$. Show u is constant.

$$E(0) = \frac{1}{2}\iint_\Omega \underbrace{u_t^2(x,y,0)}_{=0} + \underbrace{|\nabla u(x,y,0)|^2}_{u_x^2 + u_y^2 = 0 \text{ b/c } u(x,y,0)=0 \ \forall x,y \in \Omega.}$$

$= 0.$

By part b, we have $e^{-2kt}E(t) \leq e^{-2k \cdot 0} \cdot E(0) = 0$, but $E(t) \geq 0$.

Hence, $E(t) = 0 \Rightarrow u_t = u_x = u_y \equiv 0$. So, u is constant.

7. (10 points) Only work on this question if you are taking the class for a letter grade.

Let u be a solution to $\Delta u = 0$ on $\mathbf{R}^2$ such that u is constant on $\sqrt{|x|} + \sqrt{|y|} = r$ for each $r > 0$.

Prove that u is constant on $\mathbf{R}^2$.

8. (10 points) Only work on this question if you are taking the class for a letter grade.

Solve the following equation using separation of variables:

$$u_{xx} + u_{yy} = 0 \quad \text{on} \quad (0,\pi)\times(0,\pi)$$

with the boundary conditions $u(x,0)$, $u(x,\pi)$, $u(0,y)$ are all zeros for $0 \le x, y \le \pi$, but $u(\pi,y) = g(y)$ for a given continuous function $g(y)$ such that $g(0) = g(\pi) = 0$.

Using separation of variables, we assume that $u(x,y) = X(x)\cdot Y(y)$.

Then, $X''(x)\cdot Y(y) + X(x)\cdot Y''(y) = 0$ on $(0,\pi)\times(0,\pi)$.

So, we get $\frac{X''}{X}(x) = -\frac{Y''}{Y}(y)$ which need to be constant.

According to the boundary condition: $Y(0) = Y(\pi) = 0$. We know that this can happen only when the constant is positive (sin and cos).

Let $\lambda > 0$ be the constant.

Then, $Y(y) = C_1\cos(\sqrt{\lambda}y) + C_2\sin(\sqrt{\lambda}y)$.

$X(x) = d_1 e^{-\sqrt{\lambda}x} + d_2\cdot e^{\sqrt{\lambda}x}$.

$Y(0) = Y(\pi) = 0$ condition tells us that $C_1 = 0$ and $\sqrt{\lambda}$ is a natural number. So, we can define $Y_n(y) = \sin ny$ and consider $d_1\cdot e^{-nx} + d_2\cdot e^{nx}$ for $n \ge 0$.

Now, using the boundary condition, we have $X(0) = 0$ which implies $d_1 = -d_2$. So, let $X_n(x) = e^{nx} - e^{-nx}$.

$$\Rightarrow u(x,y) = \sum_{n=1}^{\infty} C_n\cdot(e^{nx} - e^{-nx})\cdot\sin ny.$$

The way to find C_n: $g(y) = u(\pi,y) = \sum_{n=1}^{\infty} C_n\cdot(e^{n\pi} - e^{-n\pi})\cdot\sin ny$

So, (using Fourier sine series) $C_n = \frac{1}{e^{n\pi} - e^{-n\pi}}\cdot\frac{2}{\pi}\int_0^{\pi} g(y)\sin ny\, dy$.

9. (10 points) Only work on this question if you are taking the class for a letter grade.

Let u be harmonic on a bounded, simply connected domain $\Omega \subset \mathbf{R}^2$.

Find all functions $F : \mathbf{R} \to \mathbf{R}$ that satisfy

$$u = F\left(\frac{y}{x}\right)$$

for all $(x, y) \in \Omega$.