Name: ______________________________ Student ID #: ________________________

This exam has 7 pages, 9 questions, and a total of **100** points.

1. I am taking the class for a letter grade:

   A. (0 points) Yes

   B. (30 points) No

2. (15 points) Find an entire function $f : \mathbf{C} \to \mathbf{C}$ such that

$$|f(3e^{it})| \leq 2$$

for all $t \in \mathbf{R}$ and

$$f(\sqrt{2} + i\sqrt{2}) = e$$

or state why no such function can exist. Make sure to justify your answer.

3. (15 points) Let $\Omega \subset \mathbf{R}^2$ be a bounded domain and $u \in C^2(\Omega)$.

   Show that u is harmonic on Ω implies u^2 is subharmonic on Ω.

u: harmonic implies that $\Delta u = u_{xx} + u_{yy} = 0$.

Now, we have
$$\begin{aligned}\Delta(u^2) &= (u^2)_{xx} + (u^2)_{yy} \\ &= (2uu_x)_x + (2uu_y)_y \\ &= 2u_x^2 + 2uu_{xx} + 2u_y^2 + 2uu_{yy} \\ &= 2(u_x^2 + u_y^2) + 2u \cdot \underbrace{(u_{xx} + u_{yy})}_{=0} \\ &= 2(u_x^2 + u_y^2).\end{aligned}$$

As $u_x^2, u_y^2 \geq 0$, we have $\Delta(u^2) \geq 0$ which implies that u^2 is subharmonic.

4. (15 points) Let $u \in C^2(\Omega)$ where $\Omega = \mathbf{R} \times (0, \infty)$. Suppose u is a solution to the initial value problem

$$\begin{aligned} u_{tt} &= u_{xx}, \quad (x,t) \in \Omega \\ u(x,0) &= \phi(x), \quad x \in \mathbf{R} \\ u_t(x,0) &= \psi(x), \quad x \in \mathbf{R}. \end{aligned}$$

   If ϕ and ψ are bounded prove or disprove that that u is bounded for all $t > 0$.

5. Consider the function $f(x) = x + \sin x$ for $x \in [0, \pi]$.

(a) (9 points) Find the Fourier cosine series of f.

The Fourier cosine series of f is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ where $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$.

- $a_0 = \frac{2}{\pi}\int_0^{\pi}(x+\sin x)\,dx = \frac{2}{\pi}\left(\frac{1}{2}x^2 - \cos x\right)\Big|_0^{\pi} = \pi + \frac{4}{\pi}$.
- $a_n = \frac{2}{\pi}\int_0^{\pi}(x+\sin x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi}(x\cos nx + \sin x\cos nx)\, dx$
  - $\int x\cos nx = \frac{1}{n}x\sin nx - \int \frac{1}{n}\sin nx = \frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx$.
  - $\int \sin x\cos nx = \int \frac{1}{2}[\sin(n+1)x + \sin((1-n)x)] \Rightarrow \begin{cases} -\frac{\cos(n+1)x}{2(n+1)} + \frac{\cos(n-1)x}{2(n-1)} & \text{if } n \neq 1. \\ \quad '' \quad + 0 & \text{if } n = 1. \end{cases}$

Therefore, $a_1 = \frac{2}{\pi}\cdot\{(-2)\} = -\frac{4}{\pi}$. For n: even number, as $n\pi - n\cdot 0$ is a multiple of 2π, we only get the latter part: $\left(\frac{1}{2(n+1)}\cdot 2 - \frac{1}{2(n-1)}\cdot 2\right)\cdot\frac{2}{\pi}$

For n: odd number, the latter part vanishes $\Rightarrow$ we get $\left(\frac{-2}{n^2}\right)\cdot\frac{2}{\pi}$.

So, the Fourier cosine series is $\frac{\pi}{2} + \frac{2}{\pi} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_n = -\frac{4}{(n^2-1)}\cdot\frac{1}{\pi}$ if n: even, $-\frac{4}{n^2}\cdot\frac{1}{\pi}$ if n: odd.

(b) (6 points) Show that the cosine series of f converges uniformly on $[0, \pi]$ without using properties of the Fourier series.

We can use Weierstrass M-test, for the a_n's we have found previously, let $f_n = a_n \cos nx$. By M-test, we only need to prove that $\sum |f_n|$ is bounded. But, $\sum |f_n| \leq \sum |a_n|$ as $|\cos nx| \leq 1$.

For the sake of simplicity let's ignore a_1. (We can do this as it is only a number that is finite.) Now, we need to prove that

$$\frac{1}{\pi}\cdot\sum_{n=2}^{\infty}\Big(\underbrace{\frac{4}{n^2-1}}_{n:\text{even}} \text{ or } \underbrace{\frac{4}{n^2}}_{n:\text{odd}}\Big) < \infty.$$

But, $\frac{4}{n^2-1} \leq \frac{8}{n^2}$ if $n \geq 2$ and $\frac{4}{n^2} < \frac{8}{n^2}$ obviously.

So, $\frac{1}{\pi}\sum_{n=2}^{\infty}\left(\frac{4}{n^2-1} \text{ or } \frac{4}{n^2}\right) < \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{8}{n^2} = \frac{8}{\pi}\cdot\sum_{n=2}^{\infty}\frac{1}{n^2} < \infty$.

$\blacksquare$

6. Let $\Omega \subset \mathbf{R}^2$ be a simply connected, bounded domain, $u \in C^2(\Omega \times \mathbf{R})$, and $c : \Omega \times \mathbf{R} \to \mathbf{R}$ is bounded by $k \in \mathbf{R}$

$$|c(x, y, t)| \leq k, \quad (x, y) \in \Omega, \ t \geq 0.$$

Suppose u is a solution of

$$\begin{aligned} u_{tt} + c(x, y, t)u_t &= \Delta u, \quad (x, y) \in \Omega, \ t > 0 \\ u(x, y, t) &= 0, \quad (x, y) \in \partial\Omega, \ t \geq 0. \end{aligned}$$

Define the mathematical energy by

$$E(t) = \frac{1}{2} \iint_\Omega u_t^2 + |\nabla u|^2 \, dA.$$

(a) (5 points) Show

$$E'(t) \leq 2kE(t).$$

(b) (3 points) Show

$$\frac{d}{dt}\left(e^{-2kt}E(t)\right) \leq 0$$

for all $t \geq 0$.

$$\frac{d}{dt}\left(e^{-2kt}E(t)\right) = -2k \cdot e^{-2kt}E(t) + e^{-2kt} \cdot E'(t)$$

$$= \underbrace{e^{-2kt}}_{>0} \cdot \underbrace{\left(E'(t) - 2kE(t)\right)}_{\leq 0 \text{ by part a.}} \leq 0.$$

(c) (2 points) Suppose $u_t(x,y,0) =$ ~~$u(x, y, 0)$~~ $= 0$ for all $(x, y) \in \Omega$. Show u is constant.

$$E(0) = \frac{1}{2}\iint_\Omega \underbrace{u_t^2(x,y,0)}_{=0} + \underbrace{|\nabla u(x,y,0)|^2}_{= u_x^2 + u_y^2 = 0 \text{ b/c } u(x,y,0)=0 \ \forall x,y \in \Omega.}$$

$= 0$.

By part b, we have $e^{-2kt}E(t) \leq e^{-2k \cdot 0} \cdot E(0) = 0$, but $E(t) \geq 0$.

Hence, $E(t) = 0 \Rightarrow u_t = u_x = u_y \equiv 0$. So, u is constant.

7. (10 points) Only work on this question if you are taking the class for a letter grade.

Let $v \in C^2(\mathbf{R}^2)$ and $\phi \in C^1(\mathbf{R})$. Suppose v solves

$$\begin{aligned} v_t &= v_{xx} + v_x^2, \quad x \in \mathbf{R},\ t \geq 0 \\ v(x,0) &= \phi(x), \quad x \in \mathbf{R}. \end{aligned}$$

Using the substitution $u = e^v$, find the fundamental solution of the above equation.

$u_t = (e^v)_t = v_t \cdot e^v$

$u_{xx} = (e^v)_{xx} = (v_x \cdot e^v)_x = v_{xx} \cdot e^v + v_x \cdot v_x \cdot e^v = (v_{xx} + v_x^2) e^v.$

Therefore, we get $u_t = u_{xx} \quad \forall x \in \mathbb{R}, t > 0.$

Moreover, $u(x,0) = e^{v(x,0)} = e^{\phi(x)} \quad \forall x \in \mathbb{R}.$

So, the fundamental solution of the above heat equation is

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t) \cdot e^{\phi(y)} dy$$

where $\Phi(x,t) = \frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{x^2}{4t}}$

$\Rightarrow$ The fundamental solution of the original problem is

$$v(x,t) = \ln\left(\int_{\mathbb{R}} \Phi(x-y,t) \cdot e^{\phi(y)} dy \right).$$

8. (10 points) Only work on this question if you are taking the class for a letter grade.

Solve the following equation using separation of variables:

$$u_{xx} + u_{yy} = 0 \quad \text{on} \quad (0,\pi)\times(0,\pi)$$

with the boundary conditions $u(x,0)$, $u(x,\pi)$, $u(0,y)$ are all zeros for $0 \le x, y \le \pi$, but $u(\pi, y) = g(y)$ for a given continuous function $g(y)$ such that $g(0) = g(\pi) = 0$.

Using separation of variables, we assume that $u(x,y) = X(x)\cdot Y(y)$.

Then, $X''(x)\cdot Y(y) + X(x)\cdot Y''(y) = 0$ on $(0,\pi)\times(0,\pi)$.

So, we get $\frac{X''}{X}(x) = -\frac{Y''}{Y}(y)$ which need to be constant.

According to the boundary condition: $Y(0) = Y(\pi) = 0$. We know that this can happen only when the constant is positive (sin and cos).

Let $\lambda > 0$ be the constant.

Then, $Y(y) = C_1 \cos(\sqrt{\lambda}\, y) + C_2 \sin(\sqrt{\lambda}\, y)$.

$X(x) = d_1 e^{-\sqrt{\lambda}\, x} + d_2 \cdot e^{\sqrt{\lambda}\, x}$.

$Y(0) = Y(\pi) = 0$ condition tells us that $C_1 = 0$ and $\sqrt{\lambda}$ is a natural number. So, we can define $Y_n(y) = \sin ny$ and consider $d_1 \cdot e^{-nx} + d_2 \cdot e^{nx}$ for $n \ge 0$.

Now, using the boundary condition, we have $X(0) = 0$ which implies $d_1 = -d_2$. So, let $X_n(x) = e^{nx} - e^{-nx}$.

$$\Rightarrow u(x,y) = \sum_{n=1}^{\infty} C_n \cdot (e^{nx} - e^{-nx}) \cdot \sin ny.$$

The way to find C_n: $g(y) = u(\pi, y) = \sum_{n=1}^{\infty} C_n \cdot (e^{n\pi} - e^{-n\pi}) \cdot \sin ny$

So, (using Fourier sine series) $C_n = \frac{1}{e^{n\pi} - e^{-n\pi}} \cdot \frac{2}{\pi} \int_0^{\pi} g(y) \sin ny \, dy.$

9. (10 points) Only work on this question if you are taking the class for a letter grade.

Let u be harmonic on a bounded, simply connected domain $\Omega \subset \mathbf{R}^2$.
Find all functions $F : \mathbf{R} \to \mathbf{R}$ that satisfy

$$u = F\left(\frac{y}{x}\right)$$

for all $(x, y) \in \Omega$.