# Quiz 9 Solutions, Sections 107—112

## True-false

**1.** Every linear operator on a nonzero vector space over $\mathbb{R}$ has at least one eigenvector.

*Solution.* **False** If all the eigenvalues of $\mathbb{R}$ are non-real numbers, then the operator will have no eigenvectors. For example, consider $L_A$ where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

□

**2.** If $\det A = 0$, then $A$ is not diagonalizable.

*Solution.* **False** Consider, for example,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

which is diagonal but has determinant 0. □

**3.** Every linear operator on a nonzero vector space over $\mathbb{C}$ has at least one eigenvalue.

*Solution.* **True** In this case $\lambda$ is an eigenvalue if and only if it is a root of the characteristic polynomial $p(t)$ of the operator. But $p(t)$ is a nonconstant polynomial with complex coefficients, so it has at least one complex root. □

**4.** Let $A \in M_{n\times n}(\mathbb{C})$. If $\dim\ker A - \lambda I \leq 1$ for every $\lambda \in \mathbb{C}$, then $A$ is diagonalizable.

*Solution.* **False** This says that the geometric multiplicity of every $\lambda$ is either 0 (if $\dim\ker A - \lambda I = 0$, that is $\lambda$ is not an eigenvalue of $A$) or 1. Thus if $A$ has any repeated eigenvalues it will *not* be diagonalizable. For example,

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

has

$$\dim\ker A - \lambda I = m_g(\lambda) = \begin{cases} 0 & \lambda \neq 2 \\ 1 & \lambda = 2 \end{cases}$$

but is not diagonalizable, because $m_a(2) = 2 > 1 = m_g(2)$. $\square$

**5.** If $A$ is similar to $B$, then $\dim\ker A - \lambda I = \dim\ker B - \lambda I$ for every $\lambda \in \mathbb{C}$.

*Solution.* **True** If $A = Q^{-1}BQ$, then

$$Q^{-1}(B - \lambda I)Q = Q^{-1}BQ - \lambda Q^{-1}IQ = Q^{-1}BQ - \lambda I = A - \lambda I,$$

so $A - \lambda I$ and $B - \lambda I$ are similar as well. It follows that they have the same rank and nullity. $\square$

**6.** If $A$ and $B$ are diagonalizable, then $AB = BA$.

*Solution.* **False** This is true if $A$ and $B$ are *simultaneously* diagonalizable, but is not true in general. For example, take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$$ $\square$

# Written

**Version 1** Consider the system of differential equations.

$$\begin{aligned} x_1'(t) &= 8x_1(t) - 6x_2(t) \\ x_2'(t) &= 9x_1(t) - 7x_2(t) \end{aligned}$$

where $x_1, x_2 : \mathbb{R} \to \mathbb{R}$ are differentiable functions. There are real numbers $\lambda_1, \lambda_2 \in \mathbb{R}$ such that every solution of this differential equation is of the form

$$\begin{aligned} x_1(t) &= a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} \\ x_2(t) &= b_1 e^{\lambda_1 t} + b_2 e^{\lambda_2 t} \end{aligned}$$

for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Compute $\lambda_1$ and $\lambda_2$.

*Solution.* Write the system as

$$A\vec{x}(t) = \vec{x}'(t), \quad \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} 8 & -6 \\ 9 & -7 \end{pmatrix}.$$

The characteristic polynomial of $A$ is

$$\begin{vmatrix} 8-\lambda & -6 \\ 9 & -7-\lambda \end{vmatrix} = -(8-\lambda)(7+\lambda) + 6\cdot 9 = -(56 - 7\lambda + 8\lambda - \lambda^2) + 54$$
$$= \lambda^2 + \lambda - 2 = (\lambda-2)(\lambda+1),$$

so we can diagonalize $A = QDQ^{-1}$ with

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

We do not actually need to compute $Q$. Setting $\vec{y}(t) = Q^{-1}\vec{x}(t)$, we see that

$$\vec{y}'(t) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \vec{y}(t) \Rightarrow \vec{y}(t) = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-t} \end{pmatrix}.$$

Therefore

$$\vec{x}(t) = Q\vec{y}(t) = \begin{pmatrix} a_1 e^{2t} + a_2 e^{-t} \\ b_1 e^{2t} + b_2 e^{-t} \end{pmatrix}$$

for some $a_i$ and $b_i$, and we conclude that $\lambda_1 = 2$, $\lambda_2 = -1$.

□

**Version 2** Let $a_n$ be a sequence of numbers defined by $a_0 = 0, a_1 = 1$, and

$$a_{n+2} = a_{n+1} + 2a_n.$$

There is an explicit formula for $a_n$ of the form

$$a_n = c_1\lambda_1^n + c_2\lambda_2^n.$$

Comptue $\lambda_1$ and $\lambda_2$ (you don't need to find the constants $c_1$ or $c_2$.)

*Solution.* We can describe this system by

$$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ a_{n+2} \end{pmatrix}.$$

In particular, we see that

$$A^n \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

The characteristic polynomial of $A$ is

$$\begin{vmatrix} -\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = -\lambda(2-\lambda) - 1 = \lambda^2 - 2\lambda - 1 = (\lambda-2)(\lambda+1),$$

so we can diagonalize $A = Q^{-1}DQ$ with

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

We do not actually need to compute $Q$. Since

$$A^n = Q^{-1} \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix} Q,$$

we get that

$$\begin{pmatrix} a_{n+1} \\ a_{n+2} \end{pmatrix} = A^n \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = Q^{-1} \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix} Q \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We see that $a_n$ must be of the form $c_1 2^n + c_2(-1)^n$, so $\lambda_1 = 2$ and $\lambda_2 = -1$. □

**Version 3** Set

$$A = \begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$$

Find an invertible matrix $Q$ and a diagonal matrix $D$ such that

$$Q^{-1}AQ = D.$$

You do not need to compute $Q^{-1}$. *Hint:* The characteristic polynomial of $A$ is $(3 - \lambda)^2(\lambda + 1)$.

*Solution.* The correct matrices are

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ \frac{3}{2} & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

To find these, we compute the characteristic polynomial of $A$ as

$$\begin{aligned} \begin{vmatrix} 7-\lambda & -4 & 0 \\ 8 & -5-\lambda & 0 \\ 6 & -6 & 3-\lambda \end{vmatrix} &= -(3-\lambda) \begin{vmatrix} 7-\lambda & -4 \\ 8 & -5-\lambda \end{vmatrix} \\ &= (\lambda - 3)\left[(7-\lambda)(-5-\lambda) + 4 \cdot 8\right] \\ &= (\lambda - 3)\left[-35 + 5\lambda - 7\lambda + \lambda^2 + 32\right] \\ &= (\lambda - 3)\left[\lambda^2 - 2\lambda - 3\lambda\right] \\ &= (\lambda - 3)(\lambda - 3)(\lambda + 1). \end{aligned}$$

We then compute eigenvectors:

$$\ker A + 1I = \ker \begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -6 & 4 \end{pmatrix} \text{ is spanned by } \begin{pmatrix} 1 \\ 2 \\ \frac{3}{2} \end{pmatrix}$$

and similarly

$$\ker A - 3I = \ker \begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix} \text{ is spanned by } \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$\square$