Quiz 8 Solutions, Sections 107—112

True-false

1. Let A be a 4×4 matrix with columns v_1, v_2, v_3, and v_4 in this order. If B is the matrix with the columns v_4, v_3, v_2, and v_1 in this order, then $\det B = \det A$.

Solution. **True** You can interchange columns twice to go from A to B: $1 \leftrightarrow 4$, $2 \leftrightarrow 3$. So there are even number of interchanges and $(-1)^{\text{even}} = 1$. □

2. Let V be an n-dimensional $\mathbb{R}$-vector space and $T : V \to V$ be a linear transformation. If T has n distinct real eigenvalues, then T^2 must have n distinct real eigenvalues.

Solution. **False** $T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. □

3. Let S and T be two linear transformations from V to V. If λ is an eigenvalue of ST, then TS also has λ as an eigenvalue.

Solution. **True** If $STv = \lambda v$, then $TSTv = T\lambda v = \lambda Tv$ so that $TS(Tv) = \lambda Tv$. (In fact, you need to consider the case $Tv = 0$. But if it were, $STv = 0$ so that $\lambda = 0$. However, as T is not invertible by having a nonzero vector v such that $Tv = 0$, TS also is not invertible and hence zero is an eigenvalue.) □

4. Let V and W be finite-dimensional vector spaces and $T : \mathcal{L}(V, W) \to \mathcal{L}(V, W)$ be a linear transformation. Then the characteristic polynomial of T in any bases is of degree $\dim V \times \dim W$.

Solution. **True** The characteristic polynomial of any $n \times n$ matrix (or equivalently, any linear transformation from an n-dimensional vector space to itself) is of degree n. □

5. Let $T : V \to W$ and $S : W \to V$ be linear transformations. If λ is an eigenvalue of ST, then TS has λ as an eigenvalue as well.

Solution. **False** Let's consider using matrices. $S = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Then $ST = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $TS = 1$. Now, 0 is an eigenvalue of ST but not of TS. $\square$

6. Let $T : V \to V$ be a linear transformation and suppose that a basis $\beta = \{v_1, \cdots, v_n\}$ of V is an eigenbasis for T. Then the corresponding eigenvalues λ_1, $\cdots$, λ_n are all distinct.

Solution. **False** Consider $T =$ the identity map and β any basis of V.

$\square$

Written

Version 1 Find two 2×2 matrices A and B that satisfy the following and show that they really do:

1) The characteristic polynomials of A and B are both $(\lambda - 1)^2$,
2) The geometric multiplicity of $\lambda = 1$ is 2 for A but 1 for B.

Solution. Let $A = I_{2\times 2}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. As they are upper triangular, we can easily see that the characteristic polynomials are $(\lambda - 1)^2$. We have $A - 1 \cdot I = 0_{2\times 2}$ so that the geometric multiplicity of $\lambda = 1$ is $\dim \ker 0_{2\times 2} = 2$. For B, we get $B - 1 \cdot I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ so that $\ker(B - I) = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ which is of 1-dimensional.

$\square$

Version 2 Find an eigenbasis of $\begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$.

Solution. Let's denote the matrix by A. Then the characteristic polynomial is

$$\det \begin{pmatrix} -\lambda & 0 & 2 \\ 1 & -\lambda & 1 \\ 0 & 1 & -2-\lambda \end{pmatrix} = (-\lambda)(-\lambda \cdot (-2-\lambda) - 1) + 2(1) = (-\lambda)(\lambda^2 + 2\lambda - 1) + 2$$

which is $-\lambda^3 - 2\lambda^2 + \lambda + 2$. Using Bezout's rational root theorem, one can find the factorization into $(1 - \lambda)(-2 - \lambda)(-1 - \lambda)$. We have three eigenvalues: $\lambda = -2$, -1, and 1. We know that the kernel of $A - \lambda I$ should be one-dimensional for those λ's because we are in $n = 3$ situation and there are three distinct eigenvalues already. It is not difficult to find $\ker(A + 2I) \ni (1, 0, -1)$, $\ker(A + I) \ni (2, -1, -1)$, and $\ker(A - I) \ni (2, 3, 1)$. Hence, $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\}$ is an eigenbasis.

□

Version 3 Let V be a 3-dimensional vector space over a field F and T, S be two linear transformations from V to V. Suppose that there exists a basis $\beta = \{v_1, v_2, v_3\}$ of V which is an eigenbasis for both T and S. Prove that $ST = TS$. [Hint. In order to prove that two linear transformations are the same, you can show that they have the same images for any elements in the domain.]

Solution. We need to show that $T(S(v)) = S(T(v))$ for all $v \in V$. In fact, as TS and ST are linear transformations and β is a basis of V, we only need to show that $T(S(v_i)) = S(T(v_i))$ for $i = 1$, 2, 3. Now, as β is an eigenbasis for T, we have λ_1, λ_2, $\lambda_3 \in F$ such that $T(v_1) = \lambda_1 v_1$, $T(v_2) = \lambda_2 v_2$, and $T(v_3) = \lambda_3 v_3$. In a similar way, we have μ_1, μ_2, $\mu_3 \in F$ such that $S(v_1) = \mu_1 v_1$, $S(v_2) = \mu_2 v_2$, and $S(v_3) = \mu_3 v_3$. Let's compute $T(S(v_i))$ and $S(T(v_i))$ separately. 1) $T(S(v_i)) = T(\mu_i v_i) = \mu_i T(v_i) = \mu_i \lambda_i v_i$. 2) $S(T(v_i)) = S(\lambda_i v_i) = \mu_i \lambda_i v_i$. Therefore, $ST = TS$. □