Quiz 7 Solutions, Sections 107—112

True-false

1. Let $A \in M_{n\times n}(\mathbb{R})$ and $\vec{b} \in \mathbb{R}^n$. If $\vec{b} \neq \vec{0}$ and there are infinitely many solutions to $A\vec{x} = \vec{b}$, then A is invertible.

Solution. **False** There are infinitely many solutions to $A\vec{x} = \vec{b}$ if and only if $\ker A$ is nontrivial if and only if A is *not* invertible. □

2. Let

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 3 & 5 \end{pmatrix}.$$

There exist two distinct matrices $B \neq B'$ such that A is row-equivalent to B and to B'.

Solution. **True** We can simply apply two different row operations to A to obtain B and B'. For example, A is row-equivalent to the two distinct matrices

$$B = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 6 & 10 \end{pmatrix} \text{ and } B' = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 12 & 20 \end{pmatrix}.$$ □

3. Let $A \in M_{m\times n}(\mathbb{R})$. Column operations on A preserve the solution space of $A\vec{x} = \vec{0}$.

Solution. **False** As a counterexample, the solutions to

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vec{x} = \vec{0} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \vec{x} = \vec{0}$$

are clearly distinct, even though these matrices are column equivalent. □

4. Let $A \in M_{n\times n}(\mathbb{R})$. If there is a unique solution to $A\vec{x} = \vec{b}$ for some $\vec{b} \in \mathbb{R}^n$, then A is invertible.

Solution. **True** If there is a unique solution, then $\ker A = \{0\}$, so A is invertible. □

5. Let $A \in M_{m\times n}(\mathbb{R})$. Row operations on A preserve the solution space of $A\vec{x} = \vec{0}$.

Solution. **True** We can think about $A\vec{x} = \vec{0}$ as corresponding to the matrix $(A|\vec{0})$ augumented by a column of zeros, and row operations will preserve this last column. □

6. Let $A \in M_{n\times n}(\mathbb{R})$. If there is a unique solution $\vec{x}$ to $A\vec{x} = \vec{0}$, then A is invertible.

Solution. **True** $\vec{x} = \vec{0}$ is always a solution, so if it is the only solution $\ker A = \{0\}$ and A is invertible. □

Written

Version 1 Find all solutions to the vector equation

$$\begin{pmatrix} 1 & 1 & -3 & 4 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

Solution. We abbreviate the above equation as $A\vec{x} = \vec{b}$. We can row-reduce the augumented matrix $(A|\vec{b})$ as

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 4 & 2 \\ 1 & 1 & 1 & -1 & 4 \\ 1 & 1 & -1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right).$$

From this, we can read off that a particular solution is given by

$$\vec{x}_0 = \begin{pmatrix} 3 \\ 0 \\ 3 \\ 2 \end{pmatrix}.$$

To find *all* solutions, we solve the associated homogeneous system. Since $\operatorname{rank} A = 3$ (there are 3 pivots) and A has 4 columns, we expect a 1-dimensional space of solutions. It is not hard to see that

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \vec{x} = 0 \Rightarrow \vec{x} = t \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

for some $t \in \mathbb{R}$. It follows that the general solution to $A\vec{x} = \vec{b}$ is

$$\begin{pmatrix} 3 \\ 0 \\ 3 \\ 2 \end{pmatrix} + \vec{x} = t \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}.$$

□

Version 2 Give an example of a matrix $A \in M_{m\times n}(\mathbb{R})$ (you can choose m and n) such that $A\vec{x} = \vec{b}$ has a solution $\vec{x}$ for every $\vec{b} \in \mathbb{R}^m$ but $A\vec{y} = \vec{0}$ has a nonzero solution $\vec{y}$. Prove that your example works.

Solution. This is equivalent to asking for a linear transformation $T : V \to W$ that is onto *and* has a nontrivial kernel. This is only possible if $\dim V > \dim W$. Returning to matrices, we see that $m > n$. One example that works is to set $m = 2, n = 1$, so that

$$A = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Then

$$A\vec{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (b)$$

has a solution $x_1 = b, x_2 = 0$ for every $b \in \mathbb{R}$. In addition, $A\vec{y} = \vec{0}$ has a nontirival solution

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0).$$

□

Version 3 Let $W = \{\vec{v} \in \mathbb{R}^4 : A\vec{v} = \vec{0}\}$, where

$$A = \begin{pmatrix} 1 & 2 & 2 & 6 \\ 3 & 6 & 5 & 17 \\ 2 & 4 & 4 & 12 \\ 2 & 4 & 0 & 8 \end{pmatrix}$$

Find a basis for W.

Solution. Row-reduce A to obtain the RREF

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that A has rank 2. Since A has 4 columns, $W = \ker A$ is a two-dimensional space. We can now read off that

$$\vec{v}_1 = \begin{pmatrix} -4 \\ 0 \\ -1 \\ 1 \end{pmatrix} \text{ and } \vec{v}_2 = \begin{pmatrix} 0 \\ -2 \\ -1 \\ 1 \end{pmatrix}$$

are elements of W. Since the vectors are linearly independent and there are 2 of them, we conclude that $\{\vec{v}_1, \vec{v}_2\}$ is a basis of W. □