Quiz 6 Solutions, Sections 107—112

True-false

1. Given the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$, let us consider another basis $\{v_1, v_2, v_3\}$ defined by $v_1 = e_1$, $v_2 = e_1 + e_2$, and $v_3 = e_1 + e_2 + e_3$. Denote by $\{f_1, f_2, f_3\}$ its dual basis. Then, for the element $f = f_1 - f_2 - f_3$, $f(3e_1 + 2e_2 + e_3) = -1$.

Solution. **True** The dual basis satisfies $f_i(v_j) = \delta_{ij}$ and note that $3e_1 + 2e_2 + e_3 = v_1 + v_2 + v_3$. So, $f(3e_1 + 2e_2 + e_3) = f(v_1 + v_2 + v_3)$ is $1 - 1 - 1 = -1$. □

2. Let T be a linear transformation from V to W (finite-dimensional ones). Choose β and γ bases of V and W respectively. Then the following formula holds:

$$[T^*]_{\beta^*}^{\gamma^*} = ([T]_\beta^\gamma)^t.$$

Solution. **False**

$$[T^*]_{\gamma^*}^{\beta^*} = ([T]_\beta^\gamma)^t$$

is the correct formula. (In fact, $[T^*]_{\beta^*}^{\gamma^*}$ does not even exist unless $V = W$.) □

3. The function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = x - y + zx$ is an element of the dual vector space $(\mathbb{R}^3)^*$.

Solution. **False** f is not linear: for example, $f(1, 0, -1) = 0$ but $f(-1, 0, 1) = -2$. □

4. Let T be a linear transformation from V to V itself (V is finite-dimensional). Given any basis β of V, the following formula must be true:

$$[T^*]_{\beta^*} = ([T]_\beta)^t.$$

Solution. **True** Apply $W = V$ and $\gamma = \beta$ to $[T^*]_{\gamma^*}^{\beta^*} = ([T]_\beta^\gamma)^t$. □

5. Let M be the 1×3 matrix $\begin{pmatrix} 1 & 7 & -3 \end{pmatrix}$ and define a function $g : \mathbb{R}^3 \to \mathbb{R}$ by $g(v) = M \cdot v$ (the matrix multiplication of 1×3 matrix M and 3×1 matrix v). Then, g is an element of the dual vector space $(\mathbb{R}^3)^*$.

Solution. **True** The dual vector space is the vector space of linear transformations from $\mathbb{R}^3$ to $\mathbb{R}$. $g(v) = Mv$ sends $\mathbb{R}^3$ vectors to $\mathbb{R}$ and by the properties of matrix multiplications we know that it is linear. □

6. Let V be a finite dimensional vector space with a basis β. For a linear transformation $T : V \to V$, the following formula is true:

$$[T^{**}]_{\beta^{**}} = [T]_\beta.$$

Solution. **True** Apply $[T^*]_{\beta^*} = ([T]_\beta)^t$ twice (one for T as it is and the other for T^*). □

Written

Version 1 Prove that if $A \in M_{2\times2}(\mathbb{R})$ satisfies $A = Q^{-1}AQ$ for all invertible 2×2 matrices Q, then $A = aI_{2\times2}$ for some scalar a. [Hint: You can let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and exploit $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or a couple of (invertible) diagonal matrices.]

Solution. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Q^{-1}AQ = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$. Comparing this with the matrix A, we get $a = d$ and $b = c$. Now, let's try $Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Then, $Q^{-1}AQ = \begin{pmatrix} a & \frac{b}{2} \\ 2c & d \end{pmatrix}$. So, $b = \frac{b}{2}$ and $c = 2c$ which imply that $b = c = 0$. Now, $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = aI_{2\times2}$. (Sanity Check: For such a matrix, we have $Q^{-1}aI_{2\times2}Q = aQ^{-1}I_{2\times2}Q = aQ^{-1}Q = aI_{2\times2}$, so it works.)

□

Version 2 Let $\beta = \left\{ \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\}$ (for simplicity, name these vectors as v_1 and v_2) be a basis of $\mathbb{R}^2$. It is known that $T(v_1) = v_1 + v_2$ and $T(v_2) = 3v_2$. Find the matrix of T in the standard basis $\mathcal{E} = \{e_1, e_2\}$, that is, find $[T]_\mathcal{E}$.

Solution. Recalling the formula

$$[I]_\beta^\mathcal{E}[T]_\beta[I]_\mathcal{E}^\beta = [T]_\mathcal{E},$$

we only need to find $[I]_\beta^\mathcal{E}$ (because for $[I]_\mathcal{E}^\beta$ you can just take the inverse). But $[I]_\beta^\mathcal{E}$ is simply $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$. Now,

$$[T]_\mathcal{E} = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -7 & 3 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 10 & -3 \\ 21 & -6 \end{pmatrix}.$$

□

Version 3 It is known that the two linear transformations defined by the condition on the left and on the right side are the same.

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Find an invertible matrix Q that shows the fact that $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ is similar to $\begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}$ and double check if the matrix gives the similarity.

Solution. Let $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ be a basis of $\mathbb{R}^2$ and $\gamma = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be another. Then, the first matrix is simply $[T]_\beta$ and the second is $[T]_\gamma$. Therefore, for $Q = [I]_\beta^\gamma$, we get $[T]_\beta = Q^{-1}[T]_\gamma Q$ so that $[T]_\beta$ is similar to $[T]_\gamma$. For $[I]_\beta^\gamma$, we need to find γ-coordinates of β vectors.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and so $Q = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$. Sanity check:

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

□