Quiz 5 Solutions, Sections 107—112

True-false

1. There exists a $n \times n$ matrix X such that for every $n \times n$ matrix A, $AX = XA$.

Solution. **True** For example, we can choose X to be the $n \times n$ identity or zero matrix. $\square$

2. Let $T : V \to V$ be a linear transformation such that $T^2 = T$. Then T must be invertible.

Solution. **False** There are many counterexamples, such as the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\vec{v}) = A\vec{v}$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. $\square$

3. There exists an invertible linear transformation $T : V \to W$ with $\dim \ker T > 0$.

Solution. **False** In order for T to be invertible, we must have $\dim \ker T = 0$. $\square$

4. Let $T : V \to W$ be a linear transformation, and $S : W \to V$ a function such that $(S \circ T)(\vec{v}) = \vec{v}$ and $(T \circ S)(\vec{w}) = \vec{w}$ for every $\vec{v} \in V$ and $\vec{w} \in W$. Then S must be linear.

Solution. **True** See Section 3.3 of Lecture Notes 11. $\square$

5. Let $T : V \to V$ be a linear transformation. Then $\dim \operatorname{Im} T \geq \dim \operatorname{Im} T^2$.

Solution. **True** $\operatorname{Im} T^2$ is a subspace of $\operatorname{Im} T$, so $\dim \operatorname{Im} T^2 \leq \dim \operatorname{Im} T$. $\square$

6. If $A \in M_{2\times 2}(\mathbb{R})$ and $A^2 = \mathcal{O}$ is the zero matrix, then $A = \mathcal{O}$.

Solution. **False** Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. $\square$

Written

Version 1 Let B be an $n \times n$, invertible matrix. Define $\Phi : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ by

$$\Phi(A) = B^{-1}AB.$$

Prove that Φ is one-to-one.

Solution. We show that Φ is linear and that it has kernel $\{0\}$, from which it follows that Φ is one-to-one.

Let $c \in \mathbb{R}$ and $A_1, A_2 \in M_{n\times n}(\mathbb{R})$. Then

$$\begin{aligned}\Phi(cA_1 + A_2) &= B^{-1}(cA_1 + A_2)B \\ &= B^{-1}(cA_1)B + B^{-1}A_2B \\ &= cB^{-1}A_1B + B^{-1}A_2B \\ &= c\Phi(A_1) + \Phi(A_2)\end{aligned}$$

so Φ is linear.

Now suppose that $\Phi(A) = 0$. Then $B^{-1}AB = 0$. If we multiply on the left by B and on the right by B^{-1}, we see that

$$BB^{-1}ABB^{-1} = B^{-1}0B.$$

But the left-hand side is $IAI = A$, and the right-hand side is 0, so $A = 0$. Thus $\ker \Phi = \{0\}$ is trivial and Φ is one-to-one. □

Version 2 Let V and W be n-dimensional vector spaces and let $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for V. Prove that a linear map $T : V \to W$ is one-to-one if $T(\beta) = \{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is a linearly independent set.

Solution. Every vector in V is a linear combination of the vectors in β, so suppose that $T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) = \vec{0}$. Then $a_1T(\vec{v}_1) + \cdots + a_nT(\vec{v}_n) = \vec{0}$. But $T(\beta)$ is independent, so each $a_i = 0$. Thus $\ker T = \{0\}$ and T is one-to-one. □

Version 3 Suppose that $T : V \to W$ is a linear transformation. Consider the related linear transformation $S : V \to \operatorname{Im} T$ defined by $S(\vec{v}) = T(\vec{v})$. Prove that S is an isomorphism if T is one-to-one.

Solution. If $S(\vec{v}) = \vec{0}$, then $T(\vec{v}) = \vec{0}$, so $\vec{v} = \vec{0}$. Thus S is one-to-one.

By definition, $\operatorname{Im} T$ is the set of $\vec{w} \in W$ with a $\vec{v} \in V$ with $T(\vec{v}) = \vec{w}$. But then $S(\vec{v}) = \vec{w}$, and this holds for every $\vec{w} \in \operatorname{Im} T$, so S is onto. As we showed before that it is one-to-one, S must be an isomorphism. □