Quiz 4 Solutions, Sections 107—112

True-false

1. Let $T : V \to W$ be a linear transformation. If $\dim V = 5$ and $\dim W = 4$, then $\ker T$ cannot be the zero vector space.

Solution. **True** By Rank-Nullity Theorem, $\dim \ker T + \dim \operatorname{im} T = 5$. If $\ker T$ is the zero vector space, we get $\dim \ker T = 0$ so that $\dim \operatorname{im} T = 5$. However, $\operatorname{im} T \subset W$ where $\dim W = 4$. So this is a contradiction. $\square$

2. Let V be a 2-dimensional vector space and $\beta = \{v, w\}$ be an ordered basis of V. Suppose that there is a linear transformation $T : V \to V$ satisfying $[T]_\beta^\beta = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Then $[T]_\gamma^\gamma = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ for the ordered basis $\gamma = \{w, v\}$.

Solution. **True** The matrix $[T]_\beta^\beta$ tells us that $T(v) = 1 \cdot v + 0 \cdot w$ and $T(w) = 0 \cdot v + 2 \cdot w$. Now, in the reverse order $\{w, v\}$, we get $T(w) = 2 \cdot w + 0 \cdot v$ and $T(v) = 0 \cdot w + 1 \cdot v$, hence the matrix $[T]_\gamma^\gamma$ written in the problem. $\square$

3. It is impossible to find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying $\ker T = \operatorname{im} T$.

Solution. **True** By Rank-Nullity Theorem, $\dim \ker T + \dim \operatorname{im} T = \dim \mathbb{R}^3 = 3$. If $\ker T = \operatorname{im} T$, then $\dim \ker T = \dim \operatorname{im} T$ but this is impossible because they should add up to 3. $\square$

4. Let $T : P_1(x) \to \mathbb{R}^2$ be a linear transformation defined by $T(p(x)) = \begin{pmatrix} p(1) \\ p'(1) \end{pmatrix}$. Let β be the ordered basis $\{x, 1\}$ of $P_1(x)$. There exists an ordered basis γ of $\mathbb{R}^2$ such that $[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Solution. **False** If there is such an ordered basis $\gamma = \{v_1, v_2\}$, $T(x) = 0 \cdot v_1 + 0 \cdot v_2 = 0$ and $T(1) = 0 \cdot v_1 + 0 \cdot v_2 = 0$. But, $T(x)$ and $T(1)$ are not zero vectors. $\square$

5. Let $T : V \to W$ be a linear transformation between $\mathbb{Q}$-vector spaces. Let $\beta = \{v_1, v_2, v_3\}$ be an ordered basis of V and $\gamma = \{w_1, w_2\}$ be an ordered basis of W. Suppose that

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

Then there exist a, b, $c \in \mathbb{Q}$ (at least one of them is nonzero) such that

$$[T(av_1 + bv_2 + cv_3)]_\gamma = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Solution. **True** One can find $(a, b, c) = (1, -2, 1)$. Or you can use Rank-Nullity Theorem: $\dim \ker T + \dim \operatorname{im} T = \dim V = 3$. For the same reason as #1 above, $\ker T$ is not the zero vector space so that it should contain a nonzero vector. As β is a basis, one can represent that nonzero vector as a (nonzero) linear combination of v_1, v_2, and v_3. □

6. Let V be a vector space of dimension n with an ordered basis β. For any linear transformation T from V to V itself, $[T]_\beta^\beta$ can never be the zero matrix.

Solution. **False** Counterexample: T is the zero linear transformation that sends every vector to the zero vector. □

Written

Version 1 Recall that the transpose of a 2×2 matrix is defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear transformation defined by

$$T(A) = \frac{A^t + A}{2}.$$

For the ordered basis $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, find $[T]_\beta^\beta$. (In this problem, you do NOT need to prove that T is linear.)

Solution. For the sake of convenience, let's denote the matrices in β by M_1, M_2, M_3, and M_4. Now, let's compute $T(M_i)$'s.

$$T(M_1) = \tfrac{M_1^t + M_1}{2} = M_1 = 1 \cdot M_1 + 0 \cdot M_2 + 0 \cdot M_3 + 0 \cdot M_4.$$

$T(M_2) = \frac{M_2^t + M_2}{2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0 \cdot M_1 + 0 \cdot M_2 + 1 \cdot M_3 + 0 \cdot M_4.$

$T(M_3) = \frac{M_3^t + M_3}{2} = M_3 = 0 \cdot M_1 + 0 \cdot M_2 + 1 \cdot M_3 + 0 \cdot M_4.$

$T(M_4) = \frac{M_4^t + M_4}{2} = M_4 = 0 \cdot M_1 + 0 \cdot M_2 + 0 \cdot M_3 + 1 \cdot M_4.$ Hence,

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

□

Version 2 Let $M_{2\times 2}(\mathbb{R})$ be the $\mathbb{R}$-vector space of 2×2 real matrices. Define a linear transformation $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by

$$T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot A \quad \text{(here } \cdot \text{ means the matrix multiplication).}$$

Let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be an ordered basis of $M_{2\times 2}(\mathbb{R})$. Compute $[T]_\beta^\beta$. (In this problem, you do NOT need to prove that T is linear.)

Solution. For the sake of convenience, let's denote the matrices in β by e_1, e_2, e_3, and e_4. Now, let's compute $T(e_i)$'s. We can actually do this in a more uniform way:

For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the result of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} c & d \\ a & b \end{pmatrix}$. Note that e_1 is where $a = 1$ and $b = c = d = 0$. So, $T(e_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = e_3$. In similar ways, we can compute $T(e_2) = e_4$, $T(e_3) = e_1$, and $T(e_4) = e_2$. Hence,

$$[T]_\beta^\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

□

Version 3 Let $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ be a linear transformation defined by

$$T(p(x)) = \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix}.$$

(a) Prove that $\beta = \{x^2 - x, x^2 + x - 2, x^2 - x - 2\} \subset P_2(\mathbb{R})$ is linearly independent.
(b) As $\dim P_2(\mathbb{R}) = 3$, we now have β as an ordered basis of $P_2(\mathbb{R})$. For the ordered basis $\gamma = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ of $\mathbb{R}^3$, compute $[T]_\beta^\gamma$.

Solution. (a) Suppose that $a(x^2-x)+b(x^2+x-2)+c(x^2-x-2)$ is the zero polynomial for some a, b, $c \in \mathbb{R}$. Then we get $(a+b+c)x^2 + (-a+b-c)x - 2(b+c) = 0$ so that $a+b+c = 0$, $-a+b-c = 0$, and $b+c = 0$. Combining the first two equations, we get $b = 0$. Now with the last one, $c = 0$ and finally $a = 0$. So the set β is linearly independent.

(b) For the sake of simplicity, denote the vectors of β by p_1, p_2, and p_3 and that of γ be w_1, w_2, and w_3. Then, $T(p_1) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 2w_3$. We can continue. $T(p_2) = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} = -2w_2$ and $T(p_3) = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} = -2w_1$. Hence,

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

□