Quiz 3 Solutions, Sections 107—112

True-false

1. Any linearly independent subset of $\mathbb{R}^3$ has at least 3 elements.

Solution. **False** Any such subset has at *most* 3 elements; $\{(1,0,0)\}$ has one element but is lienarly independent. $\square$

2. Let W_1, W_2 be subspaces of V. Then $\dim(W_1 + W_2) \leq \dim W_1 + \dim W_2$.

Solution. **True** If β_1 and β_2 are bases for W_1 and W_2, respectively, then $\beta_1 \cup \beta_2$ spans $W_1 + W_2$ and has at most $|\beta_1| + |\beta_2|$ elements. $\square$

3. Let V be a vector space of dimension 3 and $S = \{v, w\} \subseteq V$. Then S can be extended to a basis of V.

Solution. **False** If S is linearly dependent this is impossible. $\square$

4. Let W be a subspace of V. Then $\dim W \leq \dim V$.

Solution. **True** If β is a basis W, then it can be extended to a basis β' of V, and $|\beta| \leq |\beta'|$ since $\beta \subseteq \beta'$. $\square$

5. Let $S \subset V$ be a linearly independent set. Then there is a basis β of V containing S.

Solution. **True** This is a consequence of the Replacement Theorem. $\square$

6. Any linearly dependent subset of $\mathbb{R}^3$ has at most 3 elements.

Solution. **False** There are many dependent subsets with more than 3 elements: for example, $\{(x,0,0) : x \in \mathbb{R}\}$ has infinitely many elements and is dependent. $\square$

Written

Version 1 Let V be a vector space, and suppose that $\{\vec{v}, \vec{w}\} \subseteq V$ is linearly independent. Show that $\{\vec{v}, a\vec{v} + \vec{w}\}$ is linearly independent for any scalar a.

Solution. Suppose that $x_1\vec{v} + x_2(a\vec{v} + \vec{w}) = \vec{0}$ for some scalars x_1, x_2. We can rewrite this as

$$(x_1 + x_2a)\vec{v} + x_2\vec{w} = \vec{0}.$$

Because $\{\vec{v}, \vec{w}\}$ is an independent set, this implies that $(x_1 + x_2a) = 0$ and $x_2 = 0$, from which we conclude $x_1 = 0$ as well. We see that any linear combination of $\vec{v}$ and $a\vec{v} + \vec{w}$ that gives the zero vector must be trivial, so by definition $\{\vec{v}, a\vec{v} + \vec{w}\}$ is linearly independent. □

Version 2 Recall that the trace of a matrix is the sum of its diagonal entries, for example

$$\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

Let S be the subspace of $M_{2\times 2}(\mathbb{R})$ of matrices with trace zero. Find a basis for S and prove it's a basis.

Solution. It is not hard to see that every element of S can be written as

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

for some $a, b, c \in \mathbb{R}$. We guess that

$$\beta = \left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$$

is a basis of S. To prove this, we need to show that it is linearly independent, as we showed that it's spanning above. If some linear combination of these matrices is zero, we have

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

We see that $a = b = c = 0$, so this is the trivial linear combination. It follows that β is linearly independent. □

Version 3 Let $W = \{p(x) \in P_2(\mathbb{R}) : p(1) = 0\}$ be the subspace of $P_2(\mathbb{R})$ of polynomials vanishing at 1. Prove that the dimension of W is 2.

Solution. First we compute which polynomials are in W. If $p(x) = ax^2 + bx + c$ is in W,

$$p(1) = a + b + c = 0$$

We can solve for c in terms of a and b, and we see that every element of W is of the form

$$ax^2 + bx + (-a - b) = a(x^2 - 1) + b(x - 1)$$

for some $a, b \in \mathbb{R}$. We guess that $\beta = \{x^2 - 1, x - 1\}$ is a basis of W; if we can show this we are done, because β has 2 elements. From our above computation it is spanning, so we just need to show that it is independent. If some linear combination of these two polynomials gives zero, then

$$0 = a(x^2 - 1) + b(x - 1) = ax^2 + bx + (-a - b)$$

for *every* x.[1] This is only possible if $a = 0$, $b = 0$, and $-a - b = 0$. In particular, our linear combination must be trivial, so by definition β is independent. □

[1] More formally, by definition a polynomial is zero if and only if every coefficient is zero.