Quiz 2 Solutions, Sections 107—112

True-false

1. In $\mathbb{R}^3$, $(1,0,0)$ belongs to the span of $(1,1,1)$, $(2,1,3)$, and $(0,1,-1)$.

Solution. **False** In fact, $(0,1,-1) = 2(1,1,1) - (2,1,3)$, so the span of three vectors becomes the span of $(1,1,1)$ and $(2,1,3)$. There is no a and b such that $a(1,1) + b(1,3) = (0,0)$ so you cannot make $(1,0,0)$. $\square$

2. Let V be a vector space over $\mathbb{R}$. Consider three subspaces W, W_1, and W_2 satisfying

$$W \cap W_1 = W \cap W_2 = \{\vec{0}\}.$$

If $W_1 \subset W_2$, then $W \oplus W_1 = W \oplus W_2$ implies that $W_1 = W_2$.

Solution. **True** If $W_1 \neq W_2$, you can choose $w_2 \in W_2$ such that $w_2 \notin W_1$. As $W_2 \subset W \oplus W_2 = W \oplus W_1$, you can find $w \in W$ and $w_1 \in W_1$ such that $w_2 = w + w_1$. But then $w_2 - w_1 = w$ and the left hand side belongs to W_2 while the right hand side belongs to W so that $w = \vec{0}$. This implies $w_2 = w_1 \in W_1$ and it is a contradiction. $\square$

3. In $M_{2\times 2}(\mathbb{R})$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ belongs to the span of

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right\}.$$

Solution. **False** Any linear combination of these three vectors will have the same numbers for $(1,1)$-entry and $(2,2)$-entry. $\square$

4. Let V be a vector space and S_1 and S_2 be two subsets of V. If $S_1 \cap S_2 = \phi$, then

$$\operatorname{span}(S_1) \cap \operatorname{span}(S_2) = \{\vec{0}\}.$$

Solution. **False** Counterexample: $V = \mathbb{R}$, $S_1 = \{1\}$, $S_2 = \{2\}$. $S_1 \cap S_2 = \phi$, but $\operatorname{span}(S_1) = \operatorname{span}(S_2) = V$ so that the intersection is the whole V. $\square$

5. In $P_3(\mathbb{R})$, x^3+1 belongs to the span of
$$S = \{x+1, x^2+2x+1, x^3+x^2+x+1\}.$$

Solution. **True** $x^3+1 = 1\cdot(x+1) + (-1)\cdot(x^2+2x+1) + 1\cdot(x^3+x^2+x+1)$. □

6. Let V be a vector space and S_1 and S_2 be two subsets of V. Then
$$\operatorname{span}(S_1) \subset \operatorname{span}(S_2)$$
implies $S_1 \subset S_2$.

Solution. **False** Counterexample: $V = \mathbb{R}$, $S_1 = \{1\}$, $S_2 = \{2\}$. We get $\operatorname{span}(S_1) = \operatorname{span}(S_2) = V$, but $S_1 \not\subset S_2$. □

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Version 1 Let W_1 be the subspace of $P_2(\mathbb{R})$ with the defining equation $p(0) = 0$. In short, you have
$$W_1 = \{p(x) \in P_2(\mathbb{R}) : p(0) = 0\}.$$
Let W_2 be the subspace defined by
$$\{p(x) \in P_2(\mathbb{R}) : p(-1) = p(1) = 0\}.$$
Show that $P_2(\mathbb{R}) = W_1 \oplus W_2$.

Solution. We need to check two things: 1) $W_1 \cap W_2 = \{\vec{0}\}$. 2) $P_2(\mathbb{R}) = W_1 + W_2$.
1) $p(x) \in W_1 \cap W_2$ satisfies $p(0) = 0$ and $p(-1) = p(1) = 0$. This implies that $p(x)$ is divided by $x(x+1)(x-1)$, but $p(x)$ is of degree≤ 2. Therefore, $p(x) = \vec{0}$. This completes the proof.
2) $W_1 = \{ax^2 + bx : a, b \in \mathbb{R}\}$ and $W_2 = \{c(x+1)(x-1) : c \in \mathbb{R}\}$. Given $p(x) = Ax^2 + Bx + C \in P_2(\mathbb{R})$, you have $(A+C)x^2 + Bx \in W_1$ and $(-C)(x+1)(x-1) \in W_2$. They add up to $p(x)$. □

Version 2 Let W_1 be the subspace of $P_2(\mathbb{R})$ with the defining equation $p(x) = p(-x)$. In short, you have
$$W_1 = \{p(x) \in P_2(\mathbb{R}) : p(x) = p(-x)\}.$$
Find a subspace W_2 of $P_2(\mathbb{R})$ satisfying $P_2(\mathbb{R}) = W_1 \oplus W_2$. Explain why your answer works.

Solution. Let W_2 be the span of $\{x\}$, or equivalently, $W_2 = \{ax : a \in \mathbb{R}\}$. Considering $W_1 \cap W_2$, we know that $p(x) = ax$ satisfies $p(x) = p(-x)$ only if $a = 0$, hence $W_1 \cap W_2 = \{\vec{0}\}$. Now, let's consider a more explicit description of W_1: $p(x) = ax^2 + bx + c$ satisfies $p(x) = p(-x)$ if and only if $b = 0$. Hence, W_1 can be written as $\{ax^2 + c : a, c \in \mathbb{R}$. Now, given any vector $Ax^2 + Bx + C$ of $P_2(\mathbb{R})$, we have $Ax^2 + C \in W_1$ and $Bx \in W_2$ and they add up to our vector. □

Version 3 In $M_{2\times 2}(\mathbb{R})$, let

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Find two vectors u and v in $M_{2\times 2}(\mathbb{R})$ making the following hold:

$$M_{2\times 2}(\mathbb{R}) = \operatorname{span}(S) \oplus \operatorname{span}(\{u, v\}).$$

(Please note that it is the direct sum($\oplus$) not just the sum($+$).)

Solution. Let $u = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. For the sake of convenience, let the two vectors of S be x and y (in the given order).
Let's first show that $M_{2\times 2}(\mathbb{R}) = \operatorname{span}(S) + \operatorname{span}(\{u, v\})$. It is enough to show that we can generate any vector by a linear combination of u, v, x, and y. Observe that $x - u = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $y - v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Now, given any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$, we can take $au + bv + c(x - u) + d(y - v)$ to generate the given vector.
We now need to show that $\operatorname{span}(S) \cap \operatorname{span}(\{u, v\}) = \{\vec{0}\}$. However,

$$\operatorname{span}(\{u, v\}) = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\} \text{ and } \operatorname{span}(S) = \left\{ \begin{pmatrix} c & d \\ d & c \end{pmatrix} : c, d \in \mathbb{R} \right\}.$$

The intersection is where $c = d = 0$ which gives the zero vector. □