Quiz 14 Solutions, Sections 107—112

True-false

1. Given an $m \times n$ matrix A and $b \in \mathbb{R}^m$, the vector in the column space of A which is closest to b is the vector v satisfying $A^*Av = A^*b$.

Solution. **False** No, it is the Av for a solution v of $A^*Av = A^*b$. □

2. Let $T : V \to V$ be a linear map on a finite-dimensional complex vector space V. If $\langle Tv, Tv\rangle = \langle v, v\rangle$ for all $v \in V$, then T is unitary.

Solution. **True** $\langle Tv, Tv\rangle = ||Tv||^2$, so this is the definition of a unitary operator. □

3. For a complex matrix A, the column space of A^*A coincides with the column space of A always.

Solution. **False** No, they only have the same dimension and the correct statement would be Col A^*A = Col A^*, not Col A. (For a counterexample, you can think about $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.) □

4. The set of $n \times n$ (real) symmetric matrices is an $\mathbb{R}$-vector space.

Solution. **True** If A and B are symmetric, then $(A + cB)^t = A^t + cB^t = A + cB$. □

5. The set of $n \times n$ Hermitian matrices is a $\mathbb{C}$-vector space.

Solution. **False** The identity matrix $\mathcal{I}_{n\times n}$ is Hermitian. But, $(i\mathcal{I}_{n\times n})^* = -i\mathcal{I}_{n\times n}$ so that $i \cdot \mathcal{I}_{n\times n}$ (a $\mathbb{C}$-multiple of a Hermitian matrix) is not Hermitian. □

6. For a linear map $T : V \to V$ on a finite dimensional $\mathbb{R}$-vector space V, suppose that

$$\langle v, (T - T^*)v\rangle = 0$$

for all $v \in V$. Then T is self-adjoint.

Solution. **False** In fact, $\langle v, (T-T^*)v\rangle = \langle v, Tv\rangle - \langle v, T^*v\rangle = \langle v, Tv\rangle - \langle Tv, v\rangle = 0$ for any T and any v because we are doing this over the real numbers. Over the complex numbers, the given equation actually implies T is self-adjoint (or Hermitian). □

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Version 1 Let $T : V \to V$ be a linear operator on a finite dimensional inner product space V. Prove that if $TT^* = 0_{\mathcal{L}(V)}$, then $T = 0_{\mathcal{L}(V)}$.

Solution. For any $w \in V$, we get $\langle T^*w, T^*w\rangle = \langle w, TT^*w\rangle = \langle w, 0\rangle = 0$. By the (strict) positiveness property of an inner product, we can conclude that $T^*w = 0$ for all $w \in V$. So, for any vector $v \in V$, we get $\langle Tv, w\rangle = \langle v, T^*w\rangle = \langle v, 0\rangle = 0$ for all $w \in V$. This implies that $Tv = 0$ for all $v \in V$ so that $T = 0_{\mathcal{L}(V)}$.

□

Version 2 Consider $P_2(\mathbb{R})$ equipped with the inner product defined by

$$\langle p(x), q(x)\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be given by $T(p(x)) = p(-x)$, or equivalently and explicitly, $T(ax^2 + bx + c) = ax^2 - bx + c$. Show that T is self-adjoint.

Solution. To show that T is self-adjoint, we need to prove that $\langle Tp, q\rangle = \langle p, Tq\rangle$ for all $p, q \in P_2(\mathbb{R})$. However, $\langle Tp, q\rangle = \langle p(-x), q(x)\rangle = p(1)q(-1)+p(0)q(0)+p(-1)q(1) = \langle p(x), q(-x)\rangle = \langle p, Tq\rangle$.

□

Version 3 Find all 2×2 real matrices which are self-adjoint and orthogonal at the same time. (If you prefer to write down specific matrices, please list (at least) four matrices with enough reasoning about how you ended up finding such examples.)

Solution. Let A be a self-adjoint and orthogonal real matrix. As $A^t = A$ and $A^tA = I$, we get $A^2 - I = 0$. Now, CHT gives that $A^2 - (\operatorname{Tr} A)A + (\det A)I = 0$. Subtracting them, we get $(\operatorname{Tr} A)A = (\det A + 1)I$. If $\operatorname{Tr} A \neq 0$, we can let $A = cI$ and plug this into $A^2 - I = 0$. This finds $c = \pm 1$. $\operatorname{Tr} A = 0$ forces $\det A = -1$. Letting $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ (A is self-adjoint), $a + c = 0$ and $ac - b^2 = -1$ so that $c = -a$ and $a^2 + b^2 = 1$. Hence, $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ for some $0 \le \theta < 2\pi$. (The simplest four are $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$.) □