Quiz 13 Solutions, Sections 107—112

True-false

1.

$$\left[\int_0^1 \cos(2\sin(4x))e^{3x^2+5}\,dx\right]^2 \leq \int_0^1 \cos(2\sin(4x))^2\,dx \int_0^1 e^{6x^2+10}\,dx$$

Solution. **True** This is the CBS inequality applied to the functions

$$f(x) = \cos(2\sin(4(x)) \text{ and } g(x) = e^{3x^2+5}$$

with respect to the integral inner product on $C[0,1]$. $\square$

2. Let $\langle\ ,\ \rangle$ be an inner product on a vector space V over $\mathbb{C}$, and let $\vec{w} \in V$. Then the function $f : V \to \mathbb{C}$ defined by $f(\vec{v}) = \langle \vec{w}, \vec{v}\rangle$ is linear.

Solution. **False** Since $f(i\vec{v}) = \langle \vec{w}, i\vec{v}\rangle = -i\langle \vec{w}, \vec{v}\rangle = -if(\vec{v})$, the function f is not linear. $\square$

3. The function $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\langle x, y\rangle = xy$ is an inner product on $\mathbb{R}$.

Solution. **True** This is just the dot product on $\mathbb{R} = \mathbb{R}^1$. $\square$

4.

$$\sqrt{\int_1^5 \left(e^{x^3} + \cos(x)\right)^2\,dx} \leq \sqrt{\int_1^5 e^{2x^3}\,dx} + \sqrt{\int_1^5 \cos(x)^2\,dx}$$

Solution. **True** This is the triangle inequality applied to the functions

$$f(x) = e^{x^3} \text{ and } g(x) = \cos(x)$$

with respect to the integral inner product on $C[0,1]$. $\square$

5. Let $\langle\ ,\ \rangle$ be an inner product on a vector space V over $\mathbb{R}$, and let $\vec{w} \in V$. Then the function $f : V \to \mathbb{R}$ defined by $f(\vec{v}) = \langle \vec{v}, \vec{w}\rangle$ is linear.

Solution. **True** Inner products over $\mathbb{R}$ are linear in their first argument, so f is linear as well. $\square$

6. For any real numbers $a < b$,

$$\left[\int_a^b x^3\,dx\right]^2 \leq \left[\int_a^b x^2\,dx\right]\left[\int_a^b x^4\,dx\right]$$

Solution. **True** This is the CBS inequality applied to the functions

$$f(x) = x \text{ and } g(x) = x^2$$

with respect to the integral inner product on $C[a, b]$. □

Written

Version 1 Let a_1 and a_2 be positive real numbers. Prove that the function $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ defined by

$$\left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle = a_1 z_1 \overline{w_1} + a_2 z_2 \overline{w_2}$$

is an inner product on $\mathbb{C}^2$.

Solution. The proof that it is linear works just as for the standard inner product: for example,

$$\begin{aligned}
\left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle &= a_1(z_1 + x_1)\overline{w_1} + a_2(z_2 + x_2)\overline{w_2} \\
&= a_1 z_1 \overline{w_1} + a_2 z_2 + \overline{w_2} + a_1 x_1 \overline{w_1} + a_2 x_2 + \overline{w_2} \\
&= \left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle.
\end{aligned}$$

To see that it is conjugate-symmetric, observe that

$$\begin{aligned}
\left\langle \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\rangle &= a_1 w_1 \overline{z_1} + a_2 w_2 \overline{z_2} \\
&= \overline{\overline{a_1 w_1} z_1 + \overline{a_2 w_2} z_2} \\
&= \overline{a_1 \overline{w_1} z_1 + a_2 \overline{w_2} z_2} \\
&= \overline{\left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle}
\end{aligned}$$

where $\overline{a_1} = a_1$ because a_1 is real. Finally, to see that the inner product is positive-definite, suppose that (z_1, z_2) is a nonzero vector in $\mathbb{C}^2$. Then

$$\left\| \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\| = a_1 z_1 \overline{z_1} + a_2 z_2 \overline{z_2} = a_1 |z_1|^2 + a_2 |z_2|^2.$$

Since a_1 and a_2 are positive and at least one of $|z_1|^2$ or $|z_2|^2$ is positive, the above sum is positive as well. □

Version 2 Consider the subspace $W = \text{span}\{(1, i, 1)\}$ of $\mathbb{C}^3$. Find orthogonal bases for W and $W^\perp$ with respect to the standard inner product on $\mathbb{C}^3$. (No need to normalize the vectors.)

Solution. By definition, $(1, i, 1)$ is a basis for W, and there are no vectors to compare against it. We now find a basis for $W^\perp$. If $(z_1, z_2, z_3) \in W^\perp$, then

$$z_1 - iz_2 + z_3 = 0.$$

It is not hard to see that a basis of the solution space of this equation is

$$\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

However, $\langle v_1, v_2 \rangle \neq 0$ so we need to adjust the basis. We compute

$$\vec{w}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -i/2 \\ -1 \end{pmatrix}$$

and conclude that

$$\{\vec{v}_1, \vec{w}_2\} = \left\{ \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -i/2 \\ -1 \end{pmatrix} \right\}$$

is an orthonormal basis of $W^\perp$. □

Version 3 Consider the function $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = x_1 x_2 - y_1 y_2.$$

Does it define an inner product on $\mathbb{R}^2$? Prove why or why not.

Solution. It does not: the vector

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is nonzero, but

$$\langle \vec{v}, \vec{v} \rangle = 1 - 1 = 0.$$

□