Quiz 12 Solutions, Sections 107—112

True-false

1. If a 2×2 matrix A satisfies $A^2 = O$, then A has to be an upper triangular matrix whose diagonal entries are 0's.

Solution. **False** $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ contains no zeros but $A^2 = O$. $\square$

2. Let $T : V \to V$ be a linear transformation whose eigenvalue is 2 only and $\dim V = 6$. Denote $T - 2\mathcal{I}$ by U and suppose that we have three linearly independent vectors v_1, v_2, and v_3 such that $U^2 v_i = 0$ but $Uv_i \neq 0$ for each $i = 1, 2, 3$. Then $\{v_1, Uv_1, v_2, Uv_2, v_3, Uv_3\}$ is a Jordan basis for T.

Solution. **False** Be careful about the order! $\{Uv_1, v_1, Uv_2, v_2, Uv_3, v_3\}$ is a Jordan basis. $\square$

3. Let A be a nonzero 2×2 matrix such that $A^2 = O$. Then the Jordan canonical form of A is unique.

Solution. **True** If $Av = \lambda v$, then $A^2 v = \lambda^2 v$ but since $A^2 = O$, $\lambda = 0$ is the unique eigenvalue. As 0 is an eigenvalue, $\dim \ker A \geq 1$, but since A is nonzero, $\operatorname{rk} A \geq 1$. By the dimension theorem, $\dim \ker A = 1$. Now, as $A^2 = O$, $\dim \ker A^2 = 2$. Hence, the dot diagram is $\begin{matrix} \bullet \\ \bullet \end{matrix}$ so that the Jordan canonical form has to be $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $\square$

4. Let $T : V \to V$ be a linear transformation and $\dim V = 8$. Suppose that the characteristic polynomial of T is $(\lambda - 3)^5 (\lambda - 5)^3$. Then $\dim(T - 5\mathcal{I})^3 = 3$.

Solution. **True** Yes, it is the dimension of the corresponding generalized eigenspace. $\square$

5. Suppose that a nonzero 3×3 matrix A satisfies $A^2 = O$. Then there is a unique Jordan canonical form (up to reordering) of A.

Solution. **True** If $Av = \lambda v$, $A^3 v = \lambda^3 v$. So, the only eigenvalue is 0. Hence, $\dim\ker A \geq 1$. On the other hand, as A is nonzero, rk $A \geq 1$ so that $\dim\ker A \leq 2$. Now, as $A^2 = O$, $\dim\ker A^2 = 3$. Therefore, the dot diagram has to be $\begin{array}{cc} \bullet & \bullet \\ \bullet & \end{array}$. It results that the Jordan canonical form (up to reordering) is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. □

6. There exists a square matrix U such that $\dim\ker U = 1$, $\dim\ker U^2 = 3$, and $\dim\ker U^3 = 4$.

Solution. **False** As $\dim\ker U = 1$, 0 is an eigenvalue. Considering the Jordan blocks corresponding to 0, we can easily see that there should be only one Jordan block corresponding to 0. Hence, U^2 would give dim ker two. In fact, if $\dim\ker U = 1$, then $\dim\ker U^k = k$ until k reaches to the stabilizing constant. □

Written

Version 1 Find a Jordan basis and the corresponding Jordan canonical form of the following matrix: (note that the characteristic polynomial is $(\lambda - 1)^4$ and, if possible, use "top-to-bottom" algorithm)

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solution. Denote by U the matrix obtained by the given matrix minus $\mathcal{I}$. Then, there are two linearly independent columns of U, so $\ker U = 4 - 2 = 2$. One can easily see that $U^2 = O_{4\times 4}$. Hence, $\ker U^2 = 4$ and the dot diagram becomes $\begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \end{array}$ so that we can apply "top-to-bottom" algorithm. We first find $\ker U$ as the span of $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$. (These two, denoted by v_1 and v_3, will be the two upper bullets.) For the lower ones, we solve the equation $Uv_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $Uv_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. One can easily see that $v_2 = (0, 0, -1, 1)$ and $v_4 = (0, 0, 1, 0)$ work. Hence, $\beta = \{v_1, v_2, v_3, v_4\}$ gives its Jordan canonical form $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

□

Version 2 Find a Jordan basis and the corresponding Jordan canonical form of the following matrix: (for the characteristic polynomial, use cofactor expansion using the second column)

$$\begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Solution. Cofactor expansion with respect to the second column gives the characteristic polynomial $-(\lambda - 2)^3$. Denoting by A the given matrix, we get $\ker(A - 2\mathcal{I}) = \text{Span}\{(1, 0, -1), (0, 1, 0)\}$. By CHT, we already know that $(A - 2\mathcal{I})^3 = O$ so that $\ker(A - 2\mathcal{I})^3$ is of dimension 3. So, $\ker(A - 2\mathcal{I})^k$ should stabilize from $k = 2$ not $k = 1$ so that the dot diagram is $\begin{array}{cc} \bullet & \bullet \\ \bullet & \end{array}$. Let's choose an arbitrary vector, denoted by v_2 for a reason that will become clear later, not in the first row of the dot diagram: $v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Then, $(A - 2\mathcal{I})v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and let's denote this by v_1. Now, v_3 will be an arbitrary one, say $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, from $\ker(A - 2\mathcal{I})$ which is not a multiple of v_1. Then, $\beta = \{v_1, v_2, v_3\}$ will give its Jordan canonical form $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

□

Version 3 Prove that any 4×4 upper triangular matrix whose diagonal entries are all zero is nilpotent. (A square matrix U is called nilpotent if $U^k = O$ for some positive integer k.)

Solution. An upper triangular matrix A sends e_1 to 0. It sends e_2 to a multiple of e_1. It sends e_3 to a linear combination of e_1 and e_2. Finally, it sends e_4 to a linear combination of e_1, e_2, and e_3. So, $\text{im } A|_{\text{Span}\{e_1, \cdots, e_i\}} \subset \text{Span}\{e_1, \cdots, e_{i-1}\}$ for $i \geq 2$ and $\text{im } A|_{\text{Span}\{e_1\}} = \mathbf{0}$. So, $\text{im } A^4 = \text{im } A^4|_{\text{Span}\{e_1,e_2,e_3,e_4\}} = \text{im } A^3|_{\text{im } A|_{\text{Span}\{e_1,e_2,e_3,e_4\}}} \subset \text{im } A^3|_{\text{Span}\{e_1,e_2,e_3\}}$. Inductively, we get that$\subset \text{im } A^2|_{\text{Span}\{e_1,e_2\}} \subset \text{im } A|_{\text{Span}\{e_1\}} = \mathbf{0}$. The image being the zero vector space implies $A^4 = O$.

Here is another simple proof that may contain information that some may find feeling not good: The characteristic polynomial of an $n \times n$ upper triangular matrix A whose diagonal entries are all zero is λ^n. By CHT, $A^n = O$, so it is nilpotent. $\square$