Quiz 11 Solutions, Sections 107—112

True-false

1. Let $T : V \to V$ be a diagonalizable linear operator. Then V is T-cyclic.

Solution. **False** As a counterexample, let $\dim V > 1$ and set $T = \mathcal{I}_V$ the identity operator. It is diagonalizable but V cannot be $\mathcal{I}_V$-cyclic, because

$$\text{Span}\{v, \mathcal{I}(v), \mathcal{I}^2(v)\} = \text{Span}\{v, v, v\} = \text{Span}\{v\} \neq V$$

for any $v \in V$. □

2. There exists an operator $T : \mathbb{C}^3 \to \mathbb{C}^3$ such that $m_a(-1) = 2$ but $(T + \mathcal{I})^2 \neq O$.

Solution. **True** The conditions force the restriction of $T + \mathcal{I} = T - (-\mathcal{I})$ to the generalized eigenspace K_{-1} to be zero, but the parts of $T + I$ acting on the other generalized eigenspaces won't be zero. Concretely, take the operator corresponding to the Jordan matrix

$$J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

We see that

$$(J + I)^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{pmatrix} \neq O.$$

□

3. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator. If $T - \lambda\mathcal{I}$ is nilpotent, then T only has one eigenvalue.

Solution. **True** The generalized eigenspace K_λ is the subspace on which $T - \lambda\mathcal{I}$ acts nilpotently. If $T - \lambda\mathcal{I}$ is nilpotent, it acts nilpotently on all vectors, so $K_\lambda = \mathbb{C}^n$, so $m_a(\lambda) = n$. Since an operator on an n-dimensional space has at most n eigenvalues (counted with multiplicity) we see that λ is the only eigenvalue. □

4. If K_λ is the generalized eigenspace of an operator T, then K_λ is T-cyclic.

Solution. **False** If $E_\lambda = K_\lambda$, then $T|_{K_\lambda} = \lambda\mathcal{I}_{K_\lambda}$, so K_λ will not be T-cyclic. More generally, this will happen whenever there is more than one Jordan block associated to λ. □

5. Every generalized eigenspace K_λ of T is T-invariant.

Solution. **True** This was a problem on the last discussion worksheet (and a theorem from class.) □

6. Let $T : V \to V$ be a linear operator and $W_1, W_2 \subseteq V$ two T-cyclic subspaces of V. Then the subspace $W_1 + W_2$ is T-cyclic.

Solution. **False** For example, take $T = \mathcal{I}$ the identity operator and $V = \mathbb{C}^2$. If W_1 is the span of the first basis vector and W_2 the span of the second, they are both cyclic, but $W_1 + W_2 = \mathbb{C}^2$ is not T-cylic. □

Written

Version 1 Let V be a vector space over $\mathbb{C}$ of dimension 3 and $T : V \to V$ be an operator with only one eigenvalue $\lambda \in \mathbb{C}$. Suppose that $(T - \lambda\mathcal{I})^2$ is nonzero and $(T - \lambda\mathcal{I})^3 = 0$. What is $\dim\ker T - \lambda\mathcal{I}$? Justify your answer with a mathematical proof.

Solution. Set $U = T - \lambda\mathcal{I}$. Since T has only one eigenvalue, $K_\lambda = V$ and U is nilpotent. By the statement of the problem, the stabilizing exponent of U is 3. This means that $\dim\ker U < \dim\ker U^2 < \dim\ker U^3 = 3$. Futhermore, $\dim\ker U \geq 1$ because λ is an eigenvalue of V. Therefore we have a list of 3 numbers between 1 and 3 that strictly increase, so they must be $1 < 2 < 3$. It follows that $\dim\ker U = \dim\ker T - \lambda\mathcal{I} = 1$.

□

Version 2 Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be a linear operator given by $T(\vec{v}) = A\vec{v}$, where

$$A = \begin{pmatrix} 4 & 9 \\ -1 & -2 \end{pmatrix}$$

Find a basis β of $\mathbb{C}^2$ such that A is in Jordan canonical form. *Hint:* the characteristic polynomial of A is $(\lambda - 1)^2$.

Solution. We first compute the geometric multiplicity:

$$A - I = \begin{pmatrix} 3 & 9 \\ -1 & 3 \end{pmatrix} \text{ is row-equivalent to } \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$

so $m_g(1) = 1$ and A is not diagonalizable. We see that a basis for E_1 is

$$\vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

However, the algebraic mutiplicity of 1 is 2, so the generalized eigenspace K_1 must be all of $\mathbb{C}^2$. To find a Jordan basis, we want a nonzero vector $\vec{w}$ with $(A - I)\vec{w} = \vec{v}$. The system row-reduces to

$$\left(\begin{array}{cc|c} 3 & 9 & 3 \\ -1 & 3 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right) \text{ so one solution is } \vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since $A\vec{v} = \vec{v}$ and $A\vec{w} = \vec{w} + \vec{v}$, we see that if

$$\beta = \left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \text{ then } [T]_\beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

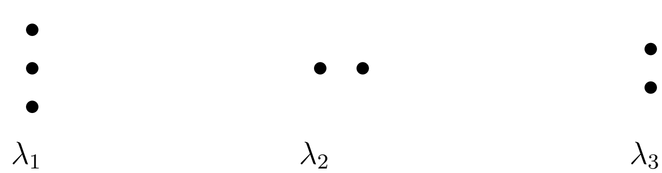
is in Jordan canonical form. (There are many other possible solutions, depending on the choice of eigenvector $\vec{v}$.) □

Version 3 Let T be an operator with Jordan canonical form

$$\begin{bmatrix} \lambda_1 & 1 & 0 & & & & \\ 0 & \lambda_1 & 1 & & & & \\ 0 & 0 & \lambda_1 & & & & \\ & & & \lambda_2 & 0 & & \\ & & & 0 & \lambda_2 & & \\ & & & & & \lambda_3 & 1 \\ & & & & & 0 & \lambda_3 \end{bmatrix}$$

where the blank areas represent blocks of zeros and λ_1, λ_2, and λ_3 are all distinct.
(a) Draw the corresponding dot diagrams for each λ_i.
(b) What is the nullity of $T - \lambda_i\mathcal{I}$ for each i?
(c) What is the nullity of $(T - \lambda_i\mathcal{I})^2$ for each i?

Solution. (a)



(b) The nullity of $T - \lambda_i \mathcal{I}$ is the number of indepedent eigenvectors, which corresponds to the first row of the dot diagrams. Therefore $\dim\ker(T - \lambda_1 \mathcal{I}) = 1$, $\dim\ker(T - \lambda_2 \mathcal{I}) = 2$, and $\dim\ker(T - \lambda_3 \mathcal{I}) = 1$.

(c) Each power of $T - \lambda_i \mathcal{I}$ annihlates one more row of the dot diagram, so the nullity of $T - \lambda_i \mathcal{I}$ is the number of dots in the first two rows of the dot diagram. Therefore $\dim\ker(T - \lambda_1 \mathcal{I})^2 = 2$, $\dim\ker(T - \lambda_2 \mathcal{I})^2 = 2$, and $\dim\ker(T - \lambda_3 \mathcal{I})^2 = 2$. $\square$