Quiz 10 Solutions, Sections 107—112

True-false

1. If J is a Jordan canonical form of a linear operator T and $c \in \mathbb{R}$, then cJ is a Jordan canonical form of the linear operator cT.

Solution. **False** A Jordan canonical form has to have 1's right above the diagonal entries, but cJ would have c's on that position. So, it is not true. □

2. $T \in M_{n\times n}(\mathbb{R})$ defines a linear transformation from $M_{n\times n}(\mathbb{R})$ to itself defined by $A \mapsto TA$. There exists a T-invariant subspace of $M_{n\times n}(\mathbb{R})$ with dimension less than or equal to n.

Solution. **True** By Cayley-Hamilton theorem, the T-cyclic subspace generated by I has dimension less than or equal to n. ($\{I, T, T^2, \cdots, T^n\}$ is linearly dependent.) □

3. The sum of two T-invariant subspaces is also T-invariant.

Solution. **True** If $T(U) \subset U$ and $T(W) \subset W$, then $T(U+W) \subset U+W$. □

4. Let T be a linear operator on a finite-dimensional vector space V whose characteristic polynomial splits, and let $\lambda_1, \cdots, \lambda_k$ be the distint eigenvalues of T. For each i, let β_i be a basis of E_{λ_i}, then $\beta_1 \cup \cdots \cup \beta_k$ is a Jordan canonical basis of T.

Solution. **False** First of all, $\beta_1 \cup \cdots \cup \beta_k$ is not even a basis unless T is diagonalizable. Second of all, even if you consider K_{λ_i} (the generalized ones), you still need to choose β_i's very carefully to get a Jordan canonical basis. □

5. Every Jordan block matrix has a unique eigenvalue.

Solution. **True** By definition, a Jordan block is the upper triangular matrix whose diagonal entries are the same numbers and there is 1 right above each diagonal entry. It is easy to see that it has only one eigenvalue. □

6. Any T-invariant subspace W of V is a T-cyclic subspace.

Solution. **False** For $\mathcal{I}_V : V \to V$, the only kind of T-cyclic subspaces is the span of one single vector. But, V is a T-invariant subspace of V. In particular, as long as $\dim V > 1$, this becomes a counterexample. □

Written

Version 1 Suppose that $A \in M_{3\times 3}(\mathbb{C})$ has only one eigenvalue 1 and the geometric multiplicity $m_g(1)$ is 2. Prove that $(A - I)^2 = O_{3\times 3}$.

Solution. We can consider all possible Jordan canonical forms J of 3×3 complex matrices with only one eigenvalue 1. We know that any similar matrices have the same geometric multiplicities, so we can compute all possible cases where $m_g(1) = 2$. Ignoring the order issue, we get

$$J = \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}.$$

You can easily compute that $(J - I)^2 = O_{3\times 3}$. However, $J = QAQ^{-1}$ for some invertible Q. Hence, $(A - I)^2 = (Q^{-1}JQ - I)^2 = Q^{-1}(J - I)^2 Q = O_{3\times 3}$.

□

Version 2 Let T be a linear operator on a finite-dimensional vector space V and W be a T-invariant subspace of V. Moreover, suppose that we have three eigenvectors v_1, v_2, and v_3 of T corresponding to all distinct eigenvalues. Prove that if $v_1 - v_2 + v_3 \in W$, then each of v_1, v_2, and v_3 belongs to W.

Solution. Let the corresponding eigenvalues of eigenvectors v_i's be λ_i's. As W is T-invariant, $T(w) \in W$ for any $w \in W$. Also, as W is a subspace, we have $\lambda_1 w \in W$. Let's pick $w = v_1 - v_2 + v_3$. Then $T(w) - \lambda_1 w \in W$ or $(T - \lambda_1 \mathcal{I}_V)(w) \in W$. In the very same way, we can apply $(T - \lambda_2 \mathcal{I}_V)$ and still get a vector in W. However, $(T - \lambda_2 \mathcal{I}_V)(T - \lambda_1 \mathcal{I}_V)(v_i)$ for $i = 1$ and $i = 2$ are zeros because v_i's are killed by $T - \lambda_i \mathcal{I}_V$. For v_3, we get

$$(T - \lambda_2 \mathcal{I}_V)(T - \lambda_1 \mathcal{I}_V)(v_3) = (T - \lambda_2 \mathcal{I}_V)(\lambda_3 - \lambda_1)(v_3) = (\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)v_3$$

and the coefficient is nonzero. So, $v_3 \in W$. In a similar way, by considering $(T - \lambda_j \mathcal{I}_V)(T - \lambda_k \mathcal{I}_V)$ for $(j, k) = (1, 3)$ and $(2, 3)$, we can deduce that v_1 and v_2 belong to W as well.

□

Version 3 Find an example that satisfies the following three conditions:

1) T is a linear operator on a finite-dimensional real vector space V and W is a T-invariant subspace of V.
2) T has some eigenvectors in V.
3) T has no eigenvectors in W.

Explain why your example satisfies the above conditions.

Solution. One can start from $T : V \to V$, but it turns out that starting from $T|_W : W \to W$ is better. What this means is that you can first find a linear transformation that does not have any eigenvectors on W and then build up V by just adding an eigenvector. Let $W = \mathbb{R}^2$ and T be a linear transformation without eigenvectors in $\mathbb{R}^2$, for example, T is the left multiplication by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Now, we just take the direct sum $W \oplus \mathbb{R}$ and define it as V and then extend T by defining T on the second component as the identity map. Then, any vector in $\mathbb{R}$ will be an eigenvector of T with the eigenvalue 1. Now 2 and 3 are satisfied. However, in fact, W is T-invariant obviously because its restriction on W was a linear operator from W to W from the beginning. Note that we can explicitly write down the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ as $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and then $W = \{(x, y, 0) : x, y \in \mathbb{R}\}$. $\square$