Quiz 1 Solutions, Sections 107—112

True-false

1. If V is a vector space over the field F and $\alpha \in F$ a non-zero scalar, then $\alpha\vec{v} = \vec{0}$ implies that $\vec{v} = \vec{0}$.

Solution. **True** This was a discussion question: the trick is to write $\vec{v} = \alpha^{-1}\alpha\vec{v}$. □

2. Let F be a field, and let f and g be two non-zero polynomials with coefficients in F. If the degree of f is n, and the degree of g is m, then the degree of $f + g$ is mn.

Solution. **False** $f(x) = x$ and $g(x) = 1 - x$ are both degree 1, but $f(x) = g(x) = 1$ is degree 0. □

3. The set of nonnegative functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space over $\mathbb{R}$ under standard operations.

Solution. **False** This set doesn't have additive inverses: for example the constant function $f(x) = 1$ is an element, but $-f(x) = -1$ is not. □

4. For any vector space V over $\mathbb{R}$ there exists exactly one element $a \in \mathbb{R}$ such that $a\vec{x} = \vec{x}$ for all $\vec{x} \in V$.

Solution. **False** For most V, this is true: the only such element is $a = 1$. But if $V = \{\vec{0}\}$ is the zero vector space, then $a\vec{0} = \vec{0}$ for every $a \in \mathbb{R}$. □

5. Every element in a field has a multiplicative inverse.

Solution. **False** $0 \in F$ does not have an inverse (but every other element does.) □

6. The set of all polynomials p for which $p(0) = 1$ is a vector space.

Solution. **False** If p and q are two such polynomials, then $(p + q)(0) = p(0) + q(0) = 1 + 1 = 2 \neq 1$, so $p + q$ is not an element of this set. For a vector space, the sum of two vectors must again be a vector. □

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Version 1 Let S contain all polynomials of the form $p(x) = a_1x + a_2x^3$ with $a_1, a_2 \in \mathbb{R}$. Let $\mathbb{R}$ be the underlying field and equip S with the standard addition and scalar multiplication operations. List at least three of the vector space properties (VS 1-8) and determine whether S satisfies each of those conditions.

Solution. S is a vector space, so it satisfies all the axioms. For completeness, we describe how to prove all of them, although you only needed to show three for full credit. Let $p(x) = a_1x + a_2x^3$ and $q(x) = b_1x + b_2x^3$ be two elements of S.

(VS 1) (commutativity of addition) By using the definition of polynomial addition,

$$\begin{aligned} p(x) + q(x) &= (a_1 + b_1)x + (a_2 + b_2)x^3 = (b_1 + a_1)x + (b_2 + a_2)x^3 \\ &= q(x) + p(x). \end{aligned}$$

(VS 2) (associativity of addition) The proof goes exactly the same as in VS 1: we use the fact that real number addition is associative.

(VS 3) The zero element is the zero polynomial $z(x) = 0x + 0x^3$, and we can check that

$$\begin{aligned} p(x) + z(x) &= (a_1 + 0)x + (a_2 + 0)x^3 = (a_1 + 0)x + (a_2 + 0)x^3 \\ &= a_1x + a_2x^3 = p(x). \end{aligned}$$

(VS 4) The additive inverse of $p(x)$ is $r(x) = -a_1x - a_2x^3$, and

$$p(x) + r(x) = (a_1 - a_1)x + (a_2 - a_2)x^3 = 0x + 0x^3 = z(x) = 0.$$

(VS 5)

$$1 \cdot p(x) = 1 \cdot (a_1x + a_2x^3) = (1 \cdot a_1)x + (1 \cdot a_2)x^3 = a_1x + a_2x^3 = p(x).$$

(VS 6) If $a, b \in \mathbb{R}$, then

$$\begin{aligned} (ab)p(x) &= (ab)(a_1x + a_2x^3) = (aba_1)x + (aba_2)x^3 \\ &= a(ba_1x + ba_2x^3) = a(bp(x)). \end{aligned}$$

(VS 7) This goes exactly as VS 6: it follows because the same property holds for the real numbers that form the coefficients of the polynomials.

(VS 8) This goes exactly as VS 6: it follows because the same property holds for the real numbers that form the coefficients of the polynomials.

□

Version 2 Let

$$\mathcal{T} = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R}\}$$

be the set of 2×2 upper-triangular matrices with real coefficients. List at least three of the vector space properties (VS 1-8) and determine whether $\mathcal{T}$ satisfies each of those properties.

Solution. $\mathcal{T}$ is a vector space, so it satisfies all the axioms. The proofs go exactly as in Version 1, except with matrix coefficients instead of polynomial coefficients. □

Version 3 Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{Q}\}$, and define an addition and scalar multiplication by

$$(a_1, a_2) + (b_1, b_2) := (a_1 + b_2, a_2 + b_1), \quad c(a_1, a_2) := (ca_1, ca_2).$$

Prove that the set V equipped with the above addition and scalar multiplication is not a vector space over $\mathbb{Q}$.

Solution. There are a number of counterexamples: here is one of them. For any vector $\vec{v}$ in any vector space, we must have

$$\vec{v} + \vec{v} = 1 \cdot \vec{v} + 1 \cdot \vec{v} = (1 + 1) \cdot \vec{v} = 2\vec{v}.$$

Set $\vec{v} = (1, 0) \in V$. Then the left-hand side of the above is

$$\vec{v} + \vec{v} = (1, 0) + (1, 0) = (1 + 0, 0 + 1) = (1, 1)$$

but the right-hand side is

$$2\vec{v} = (2, 0)$$

which is a contradiction. □