1. Let V be a finite-dimensional real vector space with inner product $\langle \cdot, \cdot \rangle$ and let $T : V \to V$ be a self-adjoint linear operator with only (strictly) positive eigenvalues. Show that the formula $(x, y) = \langle x, Ty \rangle$ defines an inner product on V.

Poll

Three things 1) Linearity in the first variable. 2) (almost) Symmetricity. 3) (Strict) Positivity.

1) $(x+x', y) = \langle x+x', Ty \rangle = \langle x, Ty \rangle + \langle x', Ty \rangle = (x,y) + (x',y)$.

2) $(y,x) = \langle y, Tx \rangle = \overline{\langle Tx, y \rangle} = \overline{\langle x, Ty \rangle} = \overline{(x,y)}$.

3) A self-adjoint operator is diagonalizable in an orthonormal basis $\{v_1, \dots, v_n\}$ (say λ_i's are corresponding eigenvalues.)

$$(x,x) = \langle x, Tx \rangle \quad (\text{let } x = a_1 v_1 + \dots + a_n v_n)$$

$$= \langle a_1 v_1 + \dots + a_n v_n, T(a_1 v_1 + \dots + a_n v_n) \rangle$$

$$= \sum_{i,j=1}^{n} \langle a_i v_i, T a_j v_j \rangle$$

$$= \sum_{i,j=1}^{n} \langle a_i v_i, \lambda_j a_j v_j \rangle$$

$$= \sum_{i,j=1}^{n} \lambda_j a_i a_j \delta_{ij} = \sum_{i=1}^{n} \lambda_i a_i^2 \geq 0 \quad \text{as } \lambda_i > 0 \text{ for all } i.$$

It is zero if and only if all of $\lambda_i a_i^2 = 0 \iff a_i = 0$ as $\lambda_i > 0$. $\iff x = \vec{0}$.

2. Let $A \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix and let λ be its eigenvalue with largest absolute value. Show that $||Ax|| \le |\lambda| \cdot ||x||$ for all $x \in \mathbb{R}^n$. Give a counterexample if A is not assumed to be symmetric.

For a real matrix, symmetric $=$ self-adjoint.

It is diagonalizable in an orthonormal basis $\{v_1, \cdots, v_n\}$ (w/ corresponding eigenvalues $\lambda_1, \cdots, \lambda_n$).

For $x = a_1v_1 + \cdots + a_nv_n$, $\quad ||x||^2 = ||a_1v_1||^2 + \cdots + ||a_nv_n||^2$ (v_i's are orthogonal to each other).

$= a_1^2 + \cdots + a_n^2$ (v_i's are of length 1 and $a_i \in \mathbb{R}$)

Now, $\quad Ax = Aa_1v_1 + \cdots + Aa_nv_n$

$= \lambda_1 a_1 v_1 + \cdots + \lambda_n a_n v_n$

so that $\quad ||Ax||^2 = (\lambda_1 a_1)^2 + \cdots + (\lambda_n a_n)^2$

$\le \lambda^2 a_1^2 + \cdots + \lambda^2 a_n^2$

$= \lambda^2 \cdot ||x||^2$

as λ is the one with the largest absolute value

$\therefore \quad ||Ax|| \le |\lambda| \cdot ||x||$.

Counterexample: A with eigenvalues 0 will give a good source since then λ would be zero.

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

1. Let $A \in M_{n\times n}(\mathbb{R})$ be a matrix that preserves orthognality in the sense that if $x \cdot y = 0$, then $Ax \cdot Ay = 0$. Show that $A = cU$ for some real number c and some orthogonal matrix U. Hint: use the orthogonality of the standard basis, and also of the pairs $\vec{e_i} + \vec{e_j}$ and $\vec{e_i} - \vec{e_j}$.

Plug in e_i, e_j combination $(i \neq j)$ $\Longrightarrow$ A's columns are orthogonal b/c Ae_i is just the i^{th} column of A.

Now, we need to show that the lengths of columns of A are the same.

Plug in e_i+e_j, e_i-e_j $(i\neq j)$ $\Longrightarrow (Ae_i + Ae_j)\cdot(Ae_i - Ae_j) = 0 \xRightarrow{\text{reduces to}} Ae_i \cdot Ae_i = Ae_j \cdot Ae_j$

b/c $Ae_i \cdot Ae_j = Ae_j \cdot Ae_i$
$(u \cdot v = v \cdot u)$.

So, let $c = \sqrt{Ae_i \cdot Ae_i}$ for any i (note that we just showed that c does not depend on the choice of i.)

Then, by what we have proven so far, $c^{-1}A = U$ for some orthogonal matrix U.

What happens for complex matrices? Instead of $Ae_i \cdot Ae_j = Ae_j \cdot Ae_i$, we have $Ae_i \cdot Ae_j = 0 = Ae_j \cdot Ae_i$.

Free Poll

1. **(True/False Jeopardy)**

(a) T F If $\vec{x}, \vec{y} \in \mathbb{R}^n$ are nonzero, then there exists an $n \times n$ orthogonal matrix A such that $A\vec{x} = \vec{y}$.

(b) T F The transpose of an orthogonal matrix is orthogonal.

(c) T F Unitary matrices are always normal.

(d) T F If $A \in M_{n\times n}(\mathbb{C})$ is symmetric, then all of its eigenvalues are real.

(e) T F The determinant of an orthogonal matrix must be 1 or -1.

(a) False.

Orthogonal matrix preserves LENGTH!

(b) True.

① Orthogonal matrix has orthonormal columns AND orthonormal rows!

② $A^tA=I$, then $A^t=A^{-1}$, so $AA^t=I$.

$$AA^t = (A^t)^t A^t$$

(c) True

If $T \cdot T^* = I \Rightarrow T^* \cdot T = I$. So, $TT^* = I = T^*T$.

(d) False

$i \cdot$ Symmetric matrix, say $\begin{pmatrix} i & \\ & i \end{pmatrix}$.

(e) True

$A^tA = I$

$$\det(A^tA) = 1.$$
$$\det(A^tA) = \det(A^t)\det(A) = \det(A)\det(A)$$

Unitary : $|\lambda| = 1$

Thank

You!